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STRUCTURE DIAGRAMS FOR PRIMITIVE BOOLEAN ALGEBRAS

JAMES WILLIAMS[1]

ABSTRACT. If S and T are structure diagrams for primitive Boolean algebras, call a homomorphism f from S onto T *right-strong* iff whenever $x\ T\ f(t)$, there is an s such that $f(s) = x$ and $s\ S\ t$; let RSE denote the category of diagrams and onto right-strong homomorphisms. The relation "S structures $\mathfrak{B}$" between diagrams and Boolean algebras induces a 1-1 correspondence between the components of RSE and the isomorphism types of primitive Boolean algebras. Up to isomorphism, each component of RSE contains a unique minimal diagram and a unique maximal tree diagram. The minimal diagrams are like those given in a construction by William Hanf. The construction which is given for producing maximal tree diagrams is recursive; as a result, every diagram S structures a Boolean algebra recursive in S.

1. **Right-strong epimorphisms.** We shall use the following notation for binary relations: If P, Q are relations, then $PQ = \{\langle x, z\rangle | \exists y: x\ P\ y$ and $y\ Q\ z\}$; $\operatorname{dom} P = \{x | \exists y: x\ P\ y\}$; $|P| = \{x | \exists y: x\ P\ y$ or $y\ P\ x\}$; $P^{\smile} = \{\langle y, x\rangle | x\ P\ y\}$; if P is a function, then $P(x) = y$ iff $x\ P\ y$; in any case $P[y] = \{x | x\ P\ y\}$. While this last condition is perhaps unconventional, it is at least compatible with the notation for Boolean algebras: $\mathfrak{A}[a] = \{b \in |\mathfrak{A}|\, | b \leq a\}$.

A Boolean algebra $\mathfrak{B}$ is *pseudo-indecomposable* iff whenever $\mathfrak{B} = \mathfrak{A} \times \mathfrak{C}$, either $\mathfrak{B} \cong \mathfrak{A}$ or $\mathfrak{B} \cong \mathfrak{C}$. A set $A \subseteq |\mathfrak{B}|$ *disjointly generates* $\mathfrak{B}$ iff each $b \in |\mathfrak{B}|$ is the sum of a pairwise disjoint finite subset of A. $\mathfrak{B}$ is *primitive* iff $\mathfrak{B}$ is pseudo-indecomposable and is disjointly generated by the set of all $b \in \mathfrak{B}$ such that $\mathfrak{B}[b]$ is pseudo-indecomposable.

William Hanf has introduced a notion of "structuring" for primitive Boolean algebras in Definition 4.1 and Lemma 4.3 of [1] which we include here with minor changes for the convenience of the present discussion:

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[1] The work for this paper was inspired by a seminar conducted by William Hanf at the University of California at Berkeley, in the Fall of 1969. At the time, the author was supported by a fellowship from the National Science Foundation.

Definition. Suppose S is a countable binary relation, $\mathfrak{B}$ is a Boolean algebra, and F is a relation whose domain is $|S|$, then S *structures* $\mathfrak{B}$ with F iff

(1) The range of F disjointly generates $\mathfrak{B}$ and contains 1 but not 0.

(2) If $a \leq b$, $s\ F\ a$, and $t\ F\ b$, then $s\ S\ t$ or $s = t$.

(3) If $a \dotplus b \leq c$ and $a, b, c \in F^{\smallsmile}[s]$, then $s\ S\ s$.

(4′) If $a \dotplus b = c$, then $F[a] \cup F[b] \supseteq F[c]$.

(5′) If $s\ S\ t$ and $t\ F\ c$, then for some $a, b \in |\mathfrak{B}|$, $s\ F\ a$, $t\ F\ b$, and $a \dotplus b = c$.

Definition 1. Like Hanf, we will call a countable transitive relation with a largest element a *diagram*. A *diagram homomorphism* $f\colon S \to T$ is a function such that $x\ S\ y$ implies $f(x)\ T\ f(y)$; f is a *right-strong* homomorphism iff whenever $x\ T\ f(t)$, there is an s such that $s\ S\ t$ and $f(s) = x$. The class of onto right-strong homomorphisms is closed under composition, so we may let RSE be the category of diagrams and onto right-strong homomorphisms.

I am indebted to the referee for pointing out the interesting work of R. S. Pierce [2]. His category $\mathfrak{Q}$, as given in Definitions 8.3 and 8.7, is similar to RSE. In particular, the category of finite diagrams and right-strong homomorphisms is essentially the category of all finite Q.O. systems which have smallest elements. For each finite diagram S, let $<_S = \{\langle y, x\rangle \mid x\ S\ y \text{ or } x = y\}$, and let $P_S = \{x \in |S| \mid x\ S\ x\}$. Then the map $S \mapsto \langle |S|, <_S, P_S\rangle$ induces the indicated isomorphism, as is easily checked.

Lemma 2. *Suppose $g\colon S \to T$ is an RSE map and S structures $\mathfrak{B}$ with F, then T structures $\mathfrak{B}$ with $g^{\smallsmile}F$.*

Proof. We need to show $g^{\smallsmile}F$ satisfies conditions (1)–(5′). Conditions (1), (2), and (4′) are straightforward. For (3), suppose that $a, b, c \in F^{\smallsmile}g[s]$ and $a \dotplus b \leq c$. Pick $u, v, w \in |S|$ so that $s\ g^{\smallsmile}u\ F$, $s\ g^{\smallsmile}v\ F\ b$, and $s\ g^{\smallsmile}w\ F\ c$. By (2) and the fact that $a, b \leq c$, we have $u\ S\ w$ or $u = w$, and $v\ S\ w$ or $v = w$. If $u\ S\ w$ or $v\ S\ w$, then $s\ T\ s$ since $s = g(u) = g(v) = g(w)$. On the other hand, if $u = v = w$, then $a, b, c \in F^{\smallsmile}[u]$ and by (3), $u\ S\ u$, and thus $s\ T\ s$. For (5′), suppose that $s\ T\ t$ and $t\ g^{\smallsmile}F\ c$; choose $v \in |S|$ so that $t\ g^{\smallsmile}\ v\ F\ c$. Since g is right-strong, we can also choose $u \in |S|$ so that $g(u) = s$ and $u\ S\ v$. Then for some $a, b \in |\mathfrak{B}|$, $u\ F\ a$, $v\ F\ b$, and $a \dotplus b = c$, by (5′) applied to S; and, of course, $s\ g^{\smallsmile}F\ a$ and $t\ g^{\smallsmile}F\ b$. □

Although the converse of the above lemma is false, the following lemma is in the same spirit, and leads to an alternate proof of the fact that every diagram structures a Boolean algebra (see Hanf's Theorem 7.2).

Definition 3. If S is a diagram, let $S^+ = S \cup \{\langle x, x\rangle \mid x \in |S|\}$, and let $\mathfrak{A}(S)$, the ideal algebra on S, be the Boolean algebra of sets generated by $\{S^+[x] \mid x \in |S|\}$. S is an *irreflexive tree* diagram iff $\forall x \in |S|$, not $x\ S\ x$, and $\forall x, y, z \in |S|$, ($x\ S\ y$ and $x\ S\ z$) imply ($y\ S^+\ z$ or $z\ S^+\ y$).

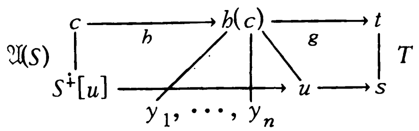
Definition 4. An RSE map $g: S \to T$ has *large collapse* iff $\forall x \in |S|$, no finite nonempty antichain A in $S[x]$ is maximal in $S[x] \cap g[g^{\vee}[A]]$.

Lemma 5. *Suppose $g: S \to T$ has large collapse and S is an irreflexive tree diagram, then T with $g^{\vee}h^{\vee}$ structures $\mathfrak{A}(S)$, where $\forall x \in |S|$, $\forall y_1, \cdots, y_n \in S[x]$, $h(S^+[x] - (S^+[y_1] \cup \cdots \cup S^+[y_n])) = x$.*

Proof. It is easy to verify that S with $h^{\vee}$ satisfies conditions (1)–(4′); thus so does T with $g^{\vee}h^{\vee}$, as in the previous lemma. For (5′), suppose $s\ T\ t$ and $t\ g^{\vee}h^{\vee}c$; then for some $y_1, \cdots, y_n \in S[h(c)]$,

$$c = S^+[h(c)] - (S^+[y_1] \cup \cdots \cup S^+[y_n]).$$

We may assume further that $\{y_1, \cdots, y_n\}$ is an antichain since S is tree ordered. We wish to choose u so that $u\ S\ h(c)$, $g(u) = s$, and u is not S-related to any y_i:



Let A be the set of y_i's for which $s\ T^+\ g(y_i)$. If A is empty, we can choose u so that $u\ S\ h(c)$, $g(u) = s$, and no y_i is below u, since S is a tree diagram and $g[s] \cap S[h(c)]$ is infinite by the strong collapse property. Such a point u will not lie below any y_i either, since otherwise $u\ S\ y_i$ implies $s\ T^+\ g(y_i)$. If A is not empty, it can be extended to an infinite antichain $A' \subseteq S[h(c)] \cap g[g^{\vee}[A]]$, by strong collapse. Since S is a tree we can choose $v \in A' - A$ so that no y_i belongs to $S[v]$. v is not below any v_i either, for suppose $v\ S^+\ y_i$: because $v \in A'$, there is some $y_j \in A$ such that $g(v) = g(y_j)$, so that $g(y_j)\ T^+\ g(y_i)$. Since $y_j \in A$, $s\ T^+\ g(y_j)$, and thus $s\ T^+\ g(y_i)$, so that $y_i \in A$; but v is not S-related to any element of A, a contradiction. Either $s = g(v)$ or $s\ T\ g(v)$ by choice of v. If $s\ T\ g(v)$, pick u so that $g(u) = s$ and $u\ S\ v$; if $s = g(v)$, let $u = v$. Then since S is a tree, u is not S-related to any y_i either, and $u\ S^+\ v\ S\ h(c)$. From this it is clear that $c = (c - S^+[u]) \dot{+} S^+[u]$, and of course $g(h(S^+[u])) = g(u) = s$, and $g(h(c - S^+[u])) = g(h(c)) = t$. □

Construction 6. *Every diagram S is the range of an RSE map μ: $MT(S) \to S$ with large collapse, where $MT(S)$ is an irreflexive tree diagram, and μ and $MT(S)$ are recursive in S.*

Proof. Let S be a diagram. Let R be the irreflexive diagram given by $|R| = \{\langle x, j\rangle | x\ S\ x,\ j \in \omega\} \cup \{\langle x, 0\rangle |$ not $x\ S\ x\}$, and $\langle x, j\rangle\ R\ \langle y, k\rangle$ iff ($x\ S\ y$ and not $y\ S\ x$) or ($x\ S\ y$ and $y\ S\ x$ and $k < j$). Let $p_1: R \to S$ be the first coordinate projection. Then R is irreflexive, and p_1 is right-strong. Let $|Q| = \{\langle 1_R, 0\rangle\} \cup (|R| - \{1_R\}) \times \omega$ and let $\langle x, j\rangle\ Q\ \langle y, k\rangle$ iff $x\ Q\ y$; let $p_2: Q \to R$ be the first coordinate projection. Then p_2 is right-strong and has large collapse. Finally, let $|MT(S)|$ be the set of all finite Q-chains with largest element 1; demand of $MT(S)$ only that whenever $\langle x_1, \cdots, x_j, \cdots, 1\rangle$ is a Q-chain, we have $\langle x_1, \cdots, 1\rangle\ MT(S)\ \langle x_j, \cdots, 1\rangle$. Let $p_3: MT(S) \to Q$ be given by $p_3(x_1, \cdots, 1) = x_1$. Then $MT(S)$ has the prescribed properties, and p_3 is right-strong. Let $\mu = p_1 p_2 p_3$; the large collapse condition carries over from p_2 to μ. Finally, it is clear from the construction that μ and $MT(S)$ are recursive in S. □

Corollary 7. *If S structures a Boolean algebra $\mathfrak{B}$, then S structures $\mathfrak{B}$ with $f^{\check{}}$ for some function f.*

Proof. S structures $\mathfrak{A}(MT(S))$ with $\mu^{\check{}} h^{\check{}}$, where h and μ are as above; by Hanf's Theorem 4.5, $\mathfrak{B} \cong \mathfrak{A}(MT(S))$. □

Corollary 8. *Every diagram S structures a Boolean algebra recursively definable in S.*

Proof. It suffices to construct an algebra isomorphic with $\mathfrak{A}(MT(S))$ that is recursive in $MT(S)$. Let $\mathbf{B}$ be the set of all pairs $\langle a, A\rangle$, where A is a finite antichain in $MT(S)[a]$. Then $\forall c \in h[MT(S)]$, $\exists ! \langle a, A\rangle \in \mathbf{B}$: $c = MT(S)^+[a] - \bigcup\{MT(S)^+[x] | x \in A\}$. Let $|\mathfrak{B}|$ be the set of all finite subsets $\{\langle a_1, A_1\rangle, \cdots, \langle a_n, A_n\rangle\}$ of B such that if for some i, j, $a_i\ MT(S)^+\ a_j$; then for some $b \in A_j$, $a_i\ MT(S)\ b$. Each element of $\mathfrak{A}(MT(S))$ is uniquely represented as the disjoint sum of elements of $h[MT(S)]$, and thus is uniquely represented by an element of $|\mathfrak{B}|$. The recursiveness in $MT(S)$ of the Boolean operations on $\mathfrak{B}$ is now straightforward. To compute the join operation, for example, first define the meet of two finite antichains A, B, by $A \wedge B = \{x \in A \cup B | x \in A \cap B$ or $\exists y \in A \cup B,\ x MT(S) y\}$. Then to find the join of a finite subset $\{\langle a_1, A_1\rangle, \cdots, \langle a_n, A_n\rangle\}$ of B, just replace each pair $\langle a_i, A_i\rangle$ and $\langle a_j, A_j\rangle$ with $\langle a_j, A_i \wedge A_j\rangle$ whenever $a_i MT(S)^+ a_j$ and $\forall b \in A_j$, not $a_i MT(S) b$. Finally, the join of two elements from $\mathfrak{B}$ is the join of their union. □

2. **Minimal and maximal diagrams.** Consider RSE to be preordered by the relation $Q \le R$ iff there is an RSE map from R to Q. The RSE

minimal diagrams turn out to be just those of the type $S(\mathfrak{B})$ given in Hanf's Construction 5.3, which for convenience, we restate as follows: Suppose $\mathfrak{B}$ is a primitive Boolean algebra. Let $PI(\mathfrak{B}) = \{b \in |\mathfrak{B}| \mid \mathfrak{B}[b]$ is pseudo-indecomposable$\}$. $\forall a \in PI(\mathfrak{B})$, let $\tau(a) = \{b \mid \mathfrak{B}[b] \cong \mathfrak{B}[a]\}$. Let $|S(\mathfrak{B})| = \{\tau(a) \mid a \in PI(\mathfrak{B})\}$, and $\forall a, b \in PI(\mathfrak{B})$, let $\tau(a)\ S(\mathfrak{B})\ \tau(b)$ iff $\mathfrak{B}[a] \times \mathfrak{B}[b] \cong \mathfrak{B}[b]$. The following result is analogous to Pierce's Lemma 9.1.

Theorem 9. *If a diagram R structures a Boolean algebra $\mathfrak{B}$, then there is a unique RSE map π from R to $S(\mathfrak{B})$.*

Proof. Assume R structures $\mathfrak{B}$ with F. We shall show that $\pi = F\tau$ is the required RSE map from R to $S(\mathfrak{B})$.

Step I. *$\pi = F\tau$ is well defined*: Suppose $s\ F\ a$ and $s\ F\ b$; then by Hanf's Lemma 4.4, $\mathfrak{B}[a] \cong \mathfrak{B}[b]$. By the proof of Hanf's Theorem 5.2, $a, b \in PI(\mathfrak{B})$, and thus $\tau(a) = \tau(b)$.

Step II. *π is onto*; that is, if $b \in PI(\mathfrak{B})$, then $\exists b' \in PI(\mathfrak{B})$, $\exists s \in |R|$, $s\ F\ b'$ and $\tau(b) = \tau(b')$: By condition (1) of the structure definition, $b = b_1 \dot{+} \cdots \dot{+} b_n$ with $b_1, \cdots, b_n$ in the range of F, and for some b_i, $\mathfrak{B}[b] \cong \mathfrak{B}[b_i]$ since $\mathfrak{B}[b]$ is pseudo-indecomposable.

Step III. *π is a homomorphism*; that is, if $s\ R\ t$, then for some $a, b \in |\mathfrak{B}|$, $s\ F\ a$, $t\ F\ b$, and $\mathfrak{B}[a] \times \mathfrak{B}[b] \cong \mathfrak{B}[b]$. Pick c so that $t\ F\ c$. Then by condition (5′) of the structure definition, we can choose a, b so that $s\ F\ a$, $t\ F\ b$, and $a \dot{+} b = c$; then by Hanf's Lemma 4.4,

$$\mathfrak{B}[b] \cong \mathfrak{B}[c] = \mathfrak{B}[a \dot{+} b] \cong \mathfrak{B}[a] \times \mathfrak{B}[b].$$

Step IV. *π is right-strong*: Suppose $\pi(s)\ S(\mathfrak{B})\ \pi(t)$. Then we may choose a, b so that $s\ F\ a$ and $t\ F\ b$; in which case $\tau(a) = \pi(s)\ S(\mathfrak{B})\ \pi(t) = \tau(b)$, so that $\mathfrak{B}[a] \times \mathfrak{B}[b] \cong \mathfrak{B}[b]$. Hence we may write $b = a_1 \dot{+} b_1$, where $\tau(a_1) = \tau(a)$ and $\tau(b_1) = \tau(b)$. By condition (1) of the structure definition we may write $a_1 = a_2 \dot{+} c$ and $b_1 = b_2 \dot{+} d$, where $\tau(a_1) = \tau(a_2)$, $\tau(b_1) = \tau(b_2)$, and a_2, b_2 are in the range of F. Choose s', t' so that $s'\ F\ a_2$ and $t'\ F\ b_2$; then $\pi(s') = \tau(a) = \pi(s)$ and $\pi(t') = \tau(b) = \pi(t)$. Since $b \geq a_2 \dot{+} b_2$, we have $(s'\ R\ t$ or $s' = t)$ and $(t'\ R\ t$ or $t' = t)$ by condition (2) of the structure definition. If $s'\ R\ t$, we are done; suppose $s' = t$. If $t' = t$, then since $b \geq a_2 \dot{+} b_2$, we have $s' = t\ R\ t$ by condition (3). If $t'\ R\ t$, however, set $b_2 = b_3 \dot{+} a_3$, with $\tau(b_3) = \tau(b_2) = \tau(b)$ and $\tau(a_3) = \tau(a)$; as before, set $a_3 = a_4 \dot{+} a_5$, with $\tau(a_3) = \tau(a_4)$, and a_4 in the range of F. Pick s'' so that $s''\ F\ a_4$. Then by condition (2), $s''\ R\ t'$ or $s'' = t'$, since $a_4 \leq b_2$. In either case $s''\ R\ t$, and $\pi(s'') = \tau(a_4) = \tau(a) = \pi(s)$.

Step V. π *is unique*: First, $\pi = F\tau$ is independent of F; that is, if R structures $\mathfrak{B}$ with F and with G, then $F\tau = G\tau$: Suppose $s\ F\ a$ and $s\ G\ b$; then by Hanf's Lemma 4.4, $\mathfrak{B}[a] \cong \mathfrak{B}[b]$, and thus $\tau(a) = \tau(b)$. Next, if $S(\mathfrak{B})$ structures $\mathfrak{B}$ with H, then $H \subseteq \tau^{\vee}$: $S(\mathfrak{B})$ structures $\mathfrak{B}$ with $\tau^{\vee}$, as is shown in the proof of Hanf's Theorem 5.4. Also, $H\tau$ is independent of H, by what we have just seen (in the special case where $R = S(\mathfrak{B})$). Hence $H\tau = \tau^{\vee}\tau$. But $\tau^{\vee}\tau$ is the identity map on $S(\mathfrak{B})$, and thus $H \subseteq \tau^{\vee}$. Finally, if g is an RSE map from R to $S(\mathfrak{B})$, then $g = \pi$: By Lemma 2, $S(\mathfrak{B})$ with $g^{\vee}F$ structures $\mathfrak{B}$. Hence $g^{\vee}F \subseteq \tau^{\vee}$, in which case $F^{\vee}g \subseteq \tau$, and thus $g \subseteq F\ F^{\vee}g \subseteq F\tau = \pi$. But g and π are functions with the same domain; so $g = \pi$. □

Corollary 10. *The* RSE *minimal diagrams are just those of the type given in Hanf's Construction* 5.3. *They are essentially the irreducible simple P.O. systems of Pierce.*

Proof. For the first statement, suppose that f: $S(\mathfrak{B}) \to R$ is an RSE map. Then R structures $\mathfrak{B}$ by Lemma 2 and Hanf's Theorem 5.4. Hence, we need only let π: $R \to S(\mathfrak{B})$ be the unique map given above. Notice that $f\pi$, being the unique RSE map from $S(\mathfrak{B})$ to $S(\mathfrak{B})$, is the identity; consequently, since f is onto, it is an isomorphism. The second statement now follows from the definitions supplied with Pierce's Propositions 8.13 and 8.14, and the fact that every right-strong homomorphism factors through an RSE map. □

Corollary 11. *The relation "S structures $\mathfrak{B}$" induces a* 1-1 *correspondence between the components of* RSE *and the isomorphism types of primitive Boolean algebras.*

Proof. If S and T structure the Boolean algebra $\mathfrak{B}$, then by the above theorem, both precede $S(\mathfrak{B})$ in RSE. Conversely, suppose f: $Q \to R$ is an RSE map. If Q structures $\mathfrak{B}$, then so does R by Lemma 2. On the other hand, if R structures $\mathfrak{B}$, then Q structures $\mathfrak{A}(MT(Q))$ by Construction 6, R structures $\mathfrak{A}(MT(Q))$ by Lemma 2, and $\mathfrak{A}(MT(Q)) \cong \mathfrak{B}$ by Hanf's Lemma 4.4; hence Q also structures $\mathfrak{B}$. A finite repetition of this argument shows that if S and T lie in the same component of RSE, they structure the same Boolean algebras. □

Lemma 12. *Suppose* R *structures* $\mathfrak{B}$, π: $R \to S(\mathfrak{B})$ *is the unique* RSE *map*, $x, y \in |R|$, *and* $R[x] \cong R[y]$ (*or more accurately*, $R|R[x] \cong R|R[y]$). *Then* $\pi(x) = \pi(y)$.

Proof. $\forall x, y \in |R|$, set $x \sim y$ iff $R[x] \cong R[y]$. Then the projection map from R to $R/\sim$ is easily seen to be right-strong. Consequently, it must factor through π. □

Theorem 13. *For any primitive Boolean algebra* $\mathfrak{B}$, *there is (up to isomorphism) a unique tree diagram* P *that structures* $\mathfrak{B}$ *and satisfies the following equivalent conditions*:

(i) *P is RSE maximal.*

(ii) *P is of the type given in Construction* 6.

(iii) *P is irreflexive, and* $\forall x \in |P|$, $P^{\vee}[x]$ *is finite and* $\forall y \in P[x]$, *there exist infinitely many* $z \in P[x]$ *such that* $P[z]$ *is a maximal ideal in* $P[x]$ *isomorphic with* $P[y]$.

Proof. The diagrams discussed will structure $\mathfrak{B}$, unless otherwise indicated. The first step is to show that if P is a tree diagram, S satisfies (iii), and $f: P \longrightarrow S$ is an RSE map, then $P \cong S$: First, P must be irreflexive since S is. Also, $\forall x$, $f|P^{\vee}[x]$ must be an embedding, and thus $P^{\vee}[x]$ is finite. For each $s \in |S|$, let $B(s)$ be the set of all $s' \in |S|$ such that $S[s'] \cong S[s]$, and s and s' have the same S-successor. For each $x \in |P|$, let $A(x)$ be the set of all $x' \in |P|$ such that $S[f(x)] \cong S[f(x')]$ and x and x' have the same P-successor. We can define an isomorphism $g: P \longrightarrow S$ inductively as follows: let $g(1) = 1$. Assume that for $n > 0$, g has been defined on $D_n = \{y | (P^{\vee}[y])^{=} = n\}$, and that for each $y \in D_n$, $S[g(y)] \cong S[f(y)]$. Extend g to each x such that $(P^{\vee}[x])^{=} = n$ as follows: Suppose y is the P-successor of x; then $f(x)\ S\ f(y)$. By (iii) we may choose s_0 so that $S[s_0] \cong S[f(x)]$ and $f(y)$ is the S-successor of s_0. By assumption, $S[f(y)] = S[g(y)]$, and thus we may choose s so that $S[s] \cong S[s_0]$ and $g(y)$ is the S-successor of s. $B(s)$ is infinite by (iii). So is $B(s_0)$; for each $s' \in B(s_0)$, there is an $x' \in P$ such that $f(x') = s'$ and $x'\ P\ y$, since $s'\ S\ f(y)$ and f is right-strong; furthermore, y is the P-successor of x' since $f(y)$ is the S-successor of s'. Each such x' belongs to $A(x)$; hence $A(x)$ is infinite. Extend g to $A(x)$ in any way such that $g|A(x)$ is a 1-1 correspondence onto $B(s)$. In this way g may be extended to the set of all x such that $(S^{\vee}[x])^{=} = n$ since both of the families $\{A(x) | x \in |P|\}$ and $\{B(s) | s \in |S|\}$ are pairwise disjoint. The completed map $q: P \longrightarrow S$ so obtained will be an isomorphism by construction.

A moment's thought shows that the diagrams given in Construction 6 satisfy (iii); thus (ii)$\longrightarrow$(iii). The above paragraph shows directly that (iii)$\longrightarrow$(i). Suppose P is maximal; then in RSE, P precedes $MT(P)$ since $MT(P)$ precedes P, and thus $P \cong MT(P)$ since $MT(P)$ satisfies (iii);

hence (i)$\to$(ii). Finally, uniqueness may be shown as follows: Suppose Q, R satisfy (iii), let $S = S(\mathfrak{B})$, and let $\pi: Q \to S$ and $\rho: R \to S$ be the unique *RSE* maps. We may define an isomorphism $\phi: Q \to R$ by a back and forth argument as follows: Since $\forall x \in |Q|$, $Q^{\vee}[x]$ is finite, we may let $\{x_i | i \in \omega\}$ be an enumeration of $|Q|$ such that $\forall x_i$, $Q^{\vee}[x_i] \subseteq \{x_j | j < i\}$. Let $\{y_i | i \in \omega\}$ be a similar enumeration of $|R|$. Let $\phi_0(1_Q) = 1_R$. Assume a partial isomorphism ϕ_j from a finite subset of $|Q|$ to $|R|$ has been defined, that for each $x \in \operatorname{dom} \phi_j$, ϕ_j maps $Q^{\vee}[x]$ onto $R^{\vee}[\phi_j(x)]$, and that $\pi(x) = \rho(\phi_j(x))$. To extend ϕ_j to the next element, say $x_i \in |Q|$ for example, let x be the Q-successor of x_i. Then $\pi(x_i)\ S\ \pi(x) = \rho(\phi_j(x))$, and we may choose $y \in |R|$ so that $y\ R\ \phi_j(x)$ and $\rho(y) = \pi(x_i)$. Then there are infinitely many points $z \in R[\phi_j(x)]$ such that $\phi_j(x)$ is the R-successor of z and $R[y] \cong R[z]$; let $\phi_{j+1}(x_i)$ be one such z that does not belong to the range of ϕ_j. Since $R[\phi_{j+1}(x_i)] \cong R[y]$, $\rho(\phi_{j+1}(x_i)) = \rho(y) = \pi(x_i)$, by the above lemma; ϕ_{j+1} maps $Q^{\vee}[x_i]$ onto $R^{\vee}[\phi_{j+1}(x_i)]$, and it is clearly a partial isomorphism. ϕ, the union of the ϕ_j's is the desired isomorphism. □

Pierce's Proposition 10.2 suggests the following analogue for diagrams and primitive Boolean algebras: Define the *product* of two diagrams S, T by $(s, t)\ S \times T\ (s', t')$ iff ($s\ S^+\ s'$ and $t\ T^+\ t'$) and ($s\ S\ s'$ or $t\ T\ t'$). For Boolean algebras $\mathfrak{A}$, $\mathfrak{B}$, let $\mathfrak{A} \oplus \mathfrak{B}$ denote the coproduct of $\mathfrak{A}$ and $\mathfrak{B}$ in the category of Boolean algebras and homomorphisms that preserve 1.

Proposition 14. *If S structures $\mathfrak{A}$ and T structures $\mathfrak{B}$, then $S \times T$ structures $\mathfrak{A} \oplus \mathfrak{B}$.*

Proof. Suppose S structures $\mathfrak{A}$ with F and T structures $\mathfrak{B}$ with G. We may consider $\mathfrak{A} \oplus \mathfrak{B}$ to consist of all formal sums $a_1 b_1 \dot{+} \cdots \dot{+} a_n b_n$ where $\forall i < j$, either $a_i a_j = 0$ or $b_i b_j = 0$, modulo the usual identities for Boolean algebras. It is now a straightforward task to show that $S \times T$, $F \times G$, and $\mathfrak{A} \oplus \mathfrak{B}$ satisfy the structure definition, except perhaps for condition (5′): Assume that $(s, t)\ S \times T\ (s', t')$ and $(s', t')\ F \times G\ a''b''$. Then $s'\ S\ a''$, $t'\ G\ b''$, and ($s\ S\ s'$ and $t\ T^+\ t'$) or vice versa. If $s\ S\ s'$ and $t = t'$, then there are $a, a' \in |\mathfrak{A}|$ such that $s\ F\ a$, $s'\ F\ a'$, and $a \dot{+} a' = a''$. But then $(s, t)\ F \times G\ (a, b'')$, $(s', t')\ F \times G\ (a', b'')$, and $ab'' \dot{+} a'b'' = a''b''$. In the other interesting case, if $s\ S\ s'$ and $t\ T\ t'$, then there are $a, a' \in |\mathfrak{A}|$ and $b, b' \in |\mathfrak{B}|$ such that $s\ F\ a$, $s'\ F\ a'$, $t\ G\ b$, $t'\ G\ b'$, $a \dot{+} a' = a''$, and $b \dot{+} b' = b''$, so that

$$ab \dot{+} a'b' \leq ab \dot{+} a'b \dot{+} ab' + a'b' = a''b'',$$

and $(s, t)\ F \times G\ ab$ and $(s', t')\ F \times G\ a'b'$; by Hanf's Lemma 4.3, this is good enough. □

The above result is interesting because it suggests that diagrams structure Boolean algebras contravariantly (as does the pairing of the embedding $S[s] \subseteq S[1]$ with the projection $\mathfrak{A} \to \mathfrak{A}[a]$, where S structures $\mathfrak{A}$ and $S|S[s]$ structures $\mathfrak{A}[a]$). As with Pierce's products of P.O. systems, the above diagram product has not yet been placed in a correct categorical framework. Perhaps there is a product-preserving contravariant functor from a category containing RSE to a suitable category of isomorphism types of Boolean algebras.

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DEPARTMENT OF MATHEMATICS, BOWLING GREEN STATE UNIVERSITY, BOWLING GREEN, OHIO 43403