AN EXTENSION OF RUNGE'S THEOREM

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ABSTRACT. Every uniformly continuously differentiable function on the compact set $X$ in the complex plane is the uniform limit of rational functions with poles off $X$.

The usual hypotheses of Runge's theorem demand that the function $f$, to be approximated, have an analytic extension to a neighborhood of the set $X$ on which the approximation is to take place. Our result only places conditions on $f$ on $X$ itself, and implies the usual form of Runge's theorem.

Definition. On a compact set $X$ in the complex plane $\mathbb{C}$, a continuous complex-valued function $f$ is said to be continuously uniformly differentiable (written $f \in D^{UC}(X)$) if there is a continuous function $f'$ on $X$ s.t. for every $\epsilon > 0$ there exists a $\delta > 0$ s.t. for every $z \in X$ and $w, w' \in X$, $w \neq w'$, with $|w - z| < \delta$, $|w' - z| < \delta$, we have

$$\left| \frac{f(w') - f(w)}{w' - w} - f'(z) \right| < \epsilon.$$  

Let $R(X)$ denote the uniform closure on $X$ of the rational functions with poles off $X$.

Theorem. $D^{UC}(X) \subseteq R(X)$.

Proof. We first observe that from (1), it follows by the Whitney extension theorem [W, p. 65] that $f$ has a $C^1$ extension (still called $f$) to all of $\mathbb{C}$. Next, it follows from (1) by an elementary argument that if $z_0$ is a point of $X$ for which there exist two sequences $\{z_n\}$ and $\{z'_n\}$ of points of $X$ that converge to $z_0$ along two different lines through $z_0$, then the Cauchy-Riemann conditions hold at $z_0$. Now we remark that $z_0$ satisfies the above condition if it is a point of density of $X$ with respect to planar Lebesgue measure $dx dy$. For if we let $E_n = \{ \theta : z_0 \pm r e^{i\theta} \notin X, 0 \leq r < 2^{-n} \}$, then all $E_n$ must have measure zero at a point of density $z_0$. But almost every point of $\{ 0 \} \times \mathbb{R}$ for which $x < 0$. Hence, $z_0$ is a point of density for $X$ by the Lebesgue density theorem.

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$X$ is a point of density of $X$. Thus, we have proved that if $f \in D^{UC}(X)$ it has a $C^1$ extension to $\mathbb{C}$ that satisfies the Cauchy-Riemann equations at almost all points of $X$. Finally, by a slight variation of the proof of [B, Corollary 3.22, p. 160], we conclude that $f \in R(X)$. Indeed, if we take $\mu \perp R(X)$ and let $\mu^\wedge$ be its Cauchy transform we have

$$\int f \, d\mu = -\frac{1}{\pi} \iint_X \frac{\partial f}{\partial z} \mu^\wedge(z) \, dx \, dy.$$ 

But $\partial f/\partial \overline{z} = 0$, a.e. on $X$ and $\mu^\wedge(z) = 0$ off $X$ so that $\mu \perp f$ and, hence, $f \in R(X)$ by the Hahn-Banach theorem.

REFERENCES
