EXISTENCE AND REPRESENTATION OF SOLUTIONS OF PARABOLIC EQUATIONS

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ABSTRACT. Let $L$ be a linear, second order parabolic operator in divergence form and let $Q$ be a bounded cylindrical domain in $E^{n+1}$. Let $\partial_p Q$ denote the parabolic boundary of $Q$. To each continuous function $f$ on $\partial_p Q$ there is a unique solution $u$ of the boundary value problem $Lu = 0$ in $Q$, $u = f$ on $\partial_p Q$. Moreover, for the given $L$ and $Q$, to each $(x, t) \in Q$ there is a unique nonnegative measure $\mu_{(x,t)}$ with support on $\partial_p Q$ such that the solution of the boundary value problem is given by $u(x, t) = \int_{\partial_p Q} f d\mu_{(x,t)}$.

I. Introduction and preliminary results. Let $\Omega \subset E^n$ be a bounded domain with compact boundary, $\partial \Omega$, and let $T > 0$. Set $Q = \Omega \times (0, T]$ and let $\partial_p Q = \partial \Omega \times [0, T] \cup \{\Omega \times (0)\}$ denote the parabolic boundary of $Q$. Write $u_{,i} = \partial u/\partial x_i$ and $u_\cdot = \partial u/\partial t$.

The given functions and solutions will lie in multidimensional $L^p$ spaces and the Sobolev space $L^2[0, T; H^{1,2}(\Omega)]$. These spaces are defined in detail by Aronson and Serrin [2]. The parabolic operator under consideration is defined by

$$Lu = u_\cdot - \{a_{ij}(x, t)u_{,i} + d_j(x, t)u_{,j} - b_j(x, t)u_j - c(x, t)u\}
$$

where products involving repeated indices $i$ or $j$ are summed for $1 \leq i, j \leq n$.

The results obtained are as follows:

Theorem 1. Let $f \in C(\partial_p Q)$. There is a unique weak solution $u$ of the boundary value problem $Lu = 0$ in $Q$, $u = f$ on $\partial_p Q$.

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Theorem 2. Let $L$ and $Q$ be given. To each $(x, t) \in Q$ there is a unique nonnegative measure $\mu_{(x, t)}$ on $\partial Q$ such that the solution $u$ corresponding to the data $f$ found in Theorem 1 is given by $u(x, t) = \int_{Q} d\mu_{(x, t)}$.

Theorem 1 is an extension of an existence result obtained by Aronson [1] restated below as Theorem B.

The coefficients appearing in the operator $L$ will be assumed to satisfy the following assumptions collectively called (H):

H.1. The $a_{ij}(x, t)$ are measurable functions in $(x, t)$ with
(a) $|a_{ij}(x, t)| \leq M < \infty$ almost everywhere in $Q$, and
(b) for some $\lambda > 0$, $a_{ij}(x, t)x_{i}x_{j} \geq \lambda |z|^{2} = \lambda \sum_{i=1}^{n} z_{i}^{2}$ for all $z \in \mathbb{R}^{n}$ and almost all $(x, t) \in Q$.

H.2. $c(x, t) \in L^{q}[0, T; L^{p}(\Omega)]$ for some pair $p, q$ satisfying

$$(*) \quad 1 < p, q \leq \infty, \quad n/2p + 1/q < 1.$$

H.3. $b_{i}(x, t), d_{i}(x, t) \in L^{q}[0, T; L^{p}(\Omega)]$ for some pair $p, q$ satisfying

$$(**) \quad 2 < p, q \leq \infty, \quad n/2p + 1/q < 1/2.$$

For easy reference one basic definition and three basic theorems are stated here without proof.

Definition 1. Let $L$ be as described as above. Assume $G(x, t) \in L^{q}[0, T; L^{p}(\Omega)]$ where $p, q$ satisfy $(*)$ and $F_{i}(x, t) \in L^{q}[0, T; L^{p}(\Omega)]$ where $p, q$ satisfy $(**)$, $u(x, t)$ is called a weak solution of the boundary value problem

\begin{equation}
Lu = G(x, t) + \{F_{i}(x, t)\}_{i} \quad \text{in } Q,
\end{equation}

\begin{equation}
\begin{aligned}
u(x, t) &= 0 \quad \text{on } S = \Omega \times [0, T], \\
u(x, t) &= u_{0}(x) \quad \text{on } \Omega
\end{aligned}
\end{equation}

if

(a) $u \in L^{2}[\delta, T; H^{1,2}_{loc}(\Omega)] \cap L^{\infty}[\delta, T; L^{2}_{loc}(\Omega)]$ for each $\delta > 0$, and
(b) $u_{0}(x) \in L^{2}(\Omega),$

and if, for each $v(x, t) \in C^{1}(Q) \cap C^{0}(\overline{Q})$ which vanishes in a neighborhood of $S$,

\begin{equation}
\int_{0}^{T} \int_{\Omega} \left[ a_{ij}u_{i}v_{j} + d_{ij}v_{j}u - b_{i}u_{i}v - cuv - uv_{i} \right] dx \, dt
\end{equation}

(c) $= \int_{0}^{T} \int_{\Omega} \left[ Gv - F_{i}v_{i} \right] dx \, dt + \int_{\Omega} u_{0}(x)v(x, 0) dx,$
and

\[(d) \lim_{t \to 0} \int_{\Omega} u(x, t)v(x, t)\,dx = \int_{\Omega} u_0(x)v(x, 0)\,dx.\]

Aronson and Serrin [2] have shown that every weak solution of (2) in $Q$ has a representative that is continuous in $Q$. Henceforth, $u$ will denote the continuous representative of a given weak solution.

**Theorem A (Maximum Principle).** Suppose $L$ satisfies (H) and let $u$ be the weak solution of $Lu = 0$ in $Q$. If $u \in C^0(\overline{Q})$ and $m_1 \leq u \leq m_2$ on $\partial P Q$, then

$$\min(m_1, 0) - Ck_1 \leq u(x, t) \leq \max(m_2, 0) + Ck_2 \quad \text{in} \quad \overline{Q}$$

where $C$ depends on $Q$ and the data in (H) and

$$k_i = \left| m_i \right| \left( \sum_{j=1}^n \| \partial_j \|_{p, q} + \| c \|_{p, q} \right) \quad \text{for} \quad i = 1, 2.$$

A proof of this theorem can be found in [2].

**Theorem B (Existence).** Suppose $L$ satisfies (H) and $u_0(x)$, $F(x, t)$, and $G(x, t)$ are as described in Definition 1. Then there is a unique weak solution $u$ of the boundary value problem (2), (3). Moreover, if $\partial \Omega$ is smooth and $u_0(x) \in C^0(\Omega)$, then $u \in C(\overline{\Omega})$.

A proof of this theorem can be found in [1].

**Theorem C (Energy Inequality).** Let $u$ be a solution of $Lu = 0$ in $Q$ with initial values $u_0 \in L^2(\Omega)$ and let $\zeta = \zeta(x)$ be a nonnegative smooth function such that $\zeta u \in L^2[0, T; H^1_0(\Omega)]$. Then there is a positive constant $C$ such that

$$\|\zeta u\|_{2, \infty}^2 + \|\zeta u_x\|_{2, 2}^2 \leq C \left\{ \int_{\Omega} \zeta^2 u_0^2\,dx + \|\zeta_x u\|_{2, 2}^2 \right\}.$$

A proof of this theorem can be found in [2]. Finally, weak solutions of $Lu = 0$ in $Q$ are locally Hölder continuous with exponent depending on the distance of the points to $\partial P Q$.

II. Existence theorem.

**Theorem 1.** Let $L$ and $Q$ be as described above. Let $f(x, t)$ be continuous on $\mathcal{S}$ and satisfy $f(x, 0) \in L^2(\Omega)$. Then there is a unique weak solution $u$ of the boundary value problem.
Proof. Note that \( f \) is continuous on \( \mathcal{S} \), a compact set; hence \( f \) can be continuously extended to \( \overline{Q} \). Let \( F(x, t) \) denote this extension. Theorem B can be used to solve the boundary value problem \( Lu = 0 \) in \( Q \), \( u(x, t) = 0 \) on \( \mathcal{S} \), \( u(x, 0) = f(x, 0) - F(x, 0) \) on \( \Omega \). Thus, the theorem will follow if the boundary value problem \( Lu = 0 \) in \( Q \), \( u(x, t) = F(x, t) \) on \( \partial_p Q \) can be solved.

For the present assume \( \partial \Omega \) is smooth. Approximate \( F \) on \( \partial_p Q \) by polynomials \( p^k(x, t) \) in the supremum norm so that on \( \partial_p Q \)

\[
m_1 = \min_{\partial_p Q} F < p^k(x, t) < \max_{\partial_p Q} F = m_2.
\]

Extend the domain of \( p^k \) to \( \overline{Q} \) so that the extension \( p^k(x, t) \in C^2(Q) \). Theorem B can be applied to solve the boundary value problem \( Lv^k = -LP^k \) in \( Q \), \( v^k = 0 \) on \( \partial_p Q \).

Define \( u^k(x, t) = v^k(x, t) + P^k(x, t) \). Then \( u^k \) satisfies

\[
Lu^k = 0 \quad \text{in } Q,
\]

\[
u(x, t) = p^k(x, t) \quad \text{on } \partial_p Q.
\]

The remainder of the proof consists of showing

(A) The solution \( u^k \) is independent of the extension \( P^k \) of \( p^k \) to \( \overline{Q} \).

(B) A subsequence of the \( u^k \) can be obtained which converges weakly in \( L^2[0, T; H^1_{\text{loc}}(\Omega)] \) for each \( \delta > 0 \) to a weak solution of \( Lu = 0 \) in \( Q \).

(C) A subsequence of that obtained in (B) converges uniformly on all compact subsets of \( Q \).

(D) The smoothness assumption on \( \partial \Omega \) is removed.

(A) Let \( P^k \) and \( \overline{P}^k \) be two extensions of \( p^k \) to \( \overline{Q} \) with \( P^k, \overline{P}^k \in C^2(Q) \) and let \( u^k, \overline{u}^k \) denote the corresponding solutions to (5). Then, since

\[
[P^k(x, t) - \overline{P}^k(x, t)] \in L^2[0, T; H^1_{\text{loc}}(\Omega)] \quad \text{and} \quad \lim_{t \to 0} [P^k(x, t) - \overline{P}^k(x, t)] = 0,
\]

it follows that \( U^k(x, t) = u^k(x, t) - \overline{u}^k(x, t) \) satisfies \( LU^k = 0 \) in \( Q \), \( U^k = 0 \) on \( \partial_p Q \). Hence, by Theorem B, \( U^k \equiv 0 \) on \( Q \). Therefore, \( P^k(x, t) \) may be assumed to be a polynomial.

(B) Since \( \partial \Omega \) is smooth and \( Lu^k = 0 \) in \( Q \), \( u^k \in C(\overline{Q}) \) and, by Theorem A,

\[
m_1 = \min(m, 0) + \bar{C}k_1 \leq u^k(x, t) \leq \max(m, 0) + \bar{C}k_2 = m_2
\]

on \( \overline{Q} \). Define
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\[ \|g\|_Q = \sup_{\delta > 0} \{ \delta \|g_x\|^2_{2,2,Q'} + \|g\|^2_{2,\infty,Q'} \}^{\frac{1}{2}} \] + \sup_Q |g| \]

where \( Q = \{x \in \Omega; \text{dist}(x, \partial \Omega) > \delta \} \times (\delta, T) \). Set \( m = \max(m_2, -m_1) \geq 0 \).

Then, by Theorem C, \( \|u^k\|_Q \leq Cm \). Hence, on each compact subcylinder \( C \) of \( Q \)

\[ \|u^k\|^2_{2,2,C} + \|u^k\|^2_{2,\infty,C} \leq \left[ \frac{Cm}{\text{dist}(C, \partial \Omega)} \right]^2. \]

Let \( \{C^j\} \) be a sequence of open cylinders with \( \overline{C^j} \subset C^{j+1} \) and \( C^j \uparrow Q \).

On \( C^1 \), the weak compactness of \( L^2[H^1,2(C)] \) and (6) imply there is a subsequence \( \{u^{1,k}\} \) of \( \{u^k\} \) which converges weakly in \( L^2[H^1,2(C^1)] \) to \( u \). Having obtained the sequence \( \{u^j,k\} \) for \( C^j \), the weak compactness of \( L^2[H^1,2(C^{j+1})] \) and (6) imply there is a subsequence \( \{u^{j+1,k}\} \) of \( \{u^j,k\} \) which converges weakly in \( L^2[H^1,2(C^{j+1})] \). Since \( \{u^{j+1,k}\} \subset \{u^j,k\} \), all of the sequences \( \{u^j,k\} \) converge weakly to \( u \) in \( L^2[H^1,2(C)] \) for any compact subcylinder \( C \) of \( Q \). Set \( u^j = u^{j+1} \). Then \( u^j \) converges weakly to \( u \) in \( L^2[H^1,2(C)] \). Hence, \( u \) satisfies \( Lu = 0 \) weakly in \( Q \) and, by Theorem C, \( \|u\|_Q \leq Cm \).

Since the \( u^j \) satisfy \( Lu = 0 \) in \( Q \), they are Hölder continuous on any cylinder \( C \) with \( \overline{C} \subset Q \). Hence, on each such cylinder, the family \( \{u^j\} \) is equicontinuous. Then, by Arzela's theorem, there is a subsequence of \( \{u^j\} \) which converges uniformly on \( C \). By using the sequence \( \{C^j\} \) given in (B) and the diagonalization process again, a subsequence of \( \{u^j\} \) is obtained which converges uniformly on any compact subset of \( Q \) to \( u \). It follows from the uniform convergence of \( p^k \) to \( F \) on \( \partial \overline{Q} \) that \( u \) is the weak solution of the boundary value problem. The uniqueness of \( u \) follows from Theorem B.

(D) Suppose \( \partial \Omega \) is not smooth. Then approximate \( \Omega \) by smooth domains \( \Omega^k \) with \( \overline{\Omega^k} \subset \Omega^{k+1} \), \( \Omega^k \uparrow \Omega \), and the argument in (C) applies to each cylinder \( Q^k = \Omega^k \times (0, T] \). Then the discussion in parts (B) and (C) can be repeated to give the unique weak solution \( u \) in \( Q \).

II. Representation theorem. In this section the following representation is obtained.

Theorem 2. Let \( L \) and \( Q \) be as described above. Then, for each \( (x, t) \in Q \), there is a unique nonnegative measure \( \mu_{(x,t)} \) concentrated on \( \partial \overline{Q} \) such that, for each continuous function \( f \) on \( \partial \overline{Q} \), the solution \( u \) of the boundary value problem (4) is given by

\[ u(x, t) = \int_{\partial \overline{Q}} f \, d\mu_{(x,t)}. \]
Moreover, for constants $a$, $A$ such that $0 < a \leq \int_{\partial_p Q} d\mu(x,t) \leq A$, it follows that the solution $u$ satisfies

$$
\min_{\partial_p Q} (a f(x,t), A f(x,t)) \leq u(x,t) \leq \max_{\partial_p Q} (a f(x,t), A f(x,t)).
$$

**Proof.** Define for each $(x,t) \in \overline{Q}$ the functional $\Lambda_{(x,t)}$ on $C(\partial_p Q)$ by

$$
\Lambda_{(x,t)} f = u(x,t) \quad \text{on} \quad Q,
$$

$$
= f(x,t) \quad \text{on} \quad \partial_p Q.
$$

$\Lambda_{(x,t)}$ is clearly a positive linear functional and the desired result follows from the Riesz representation theorem.

**REFERENCES**


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