PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 47, Number 1, January 1975

NONREALIZABILITY OF SOME CYCLIC COMPLEX BORDISM MODULES

RAPHAEL S. ZAHLER[1]

ABSTRACT. It is shown that certain modules over the complex cobordism ring MU^* cannot be realized in the sense that none of them can be isomorphic to the MU-cohomology module of a finite complex. Potential applications in stable homotopy theory are discussed.

A module M over the complex cobordism ring $MU^* = Z[x_1, x_2, \cdots]$ is said to be *realizable* if there is a finite cell complex X with $MU^*(X) = M$. The work of Larry Smith and others has shown that realizability questions are closely related to problems in the stable homotopy of spheres. In particular, if p is prime let

$$M_t = MU^*/(p^2, [CP(p-1)]^p, [V^{2p^2-2}]^{tp}),$$

where V^{2p^2-2} is the first Milnor manifold (see [6]), and consider the statement

(1) M_1 is not realizable for $p \geq 3$.

In [6] Smith used Toda's deep stable-homotopy result

$$(2) \qquad {}_p\pi^S_{p^2(2p-2)-2} = 0, \qquad p \text{ odd [8]},$$

to prove (1), and suggested that, conversely, an independent proof of (1) would yield a new proof of (2). Later [7], Smith gave an intricate proof of (1), relying not on (2), but on the mod p Hopf-invariant-one theorem. In this note we present an instant proof of (1), which moreover generalizes easily to

Theorem 1. *M_t is not realizable for t prime to p, any $p \geq 2$.*

Unfortunately, there may be a gap in efforts to derive (2) from (1).

Received by the editors February 6, 1974.

AMS (MOS) subject classifications (1970). Primary 55E45, 55G25.

Key words and phrases. Complex bordism module, realizable, BP-cohomology, invariant ideals.

[1] Partially supported by an NSF grant.

Copyright © 1975, American Mathematical Society

Ideals, invariant and otherwise. Fix a prime p (even or odd), and consider the cohomology theory given by the p-primary BP spectrum [9]. The coefficient ring BP^* is a polynomial algebra $BP^* \cong Z_{(p)}[v_1, v_2, \cdots]$.

Proposition 2. *The ideal* $J = (p^2, v_1^p, v_2^p)$ *of* BP^* *is not invariant under the action of the algebra of cohomology operations* $A = BP^*(BP)$.

Proof. Consider the operations $r_j \in A$ [9, §3]. It is easy to check via the Hazewinkel-Liulevicius formula [3], [5] that

$$r_j v_2 = \begin{cases} -v_1^p \bmod p, & j = 1; \\ v_1 \bmod p, & j = p; \\ 0 \bmod p, & \text{otherwise}. \end{cases} \tag{3}$$

Write $r_1 v_2 = x v_1^p$, $r_p v_2 = y v_1$, where x and y are units of $Z_{(p)}$. Then

$$\begin{aligned} r_p v_2^p &= p[(r_p v_2) v_2^{p-1}] + (r_1 v_2)^p \quad \text{(by the Cartan formula)} \\ &= p y v_1 v_2^{p-1} + x^p v_1^{p^2}, \end{aligned}$$

which does not lie in (p^2, v_1^p, v_2^p). This proves the proposition.

Now suppose M is an A-module, and α is a nonzero element of M such that $r_E \alpha = 0$ for any $E \neq 0$. Such an α is called *primitive*. For example, if $M^n = 0$ for $n > N$, then any nonzero element of M^N is primitive.

Corollary 3. *J cannot be the annihilator ideal of any primitive element* α.

Proof. If so, we would have $v_2^p \alpha = 0$, whence

$$0 = r_p(v_2^p \alpha) = (r_p v_2^p)\alpha + \sum_{j=0}^{p-1} (r_j v_2^p)(r_{p-j}\alpha) = (r_p v_2^p)\alpha = y p v_1 v_2^{p-1} \alpha \neq 0,$$

a contradiction. ("Annihilator ideals of primitive elements are invariant.")

Corollary 4. *There is no X such that* $BP^*(X) \cong BP^*/(p^2, v_1^p, v_2^p)$.

For J is the annihilator ideal of a generator of this cyclic BP^*-module.

Corollary 5. *$\dot{M}_1$ is not realizable.*

Proof. Tensor with $Z_{(p)}$ and apply the Quillen idempotent $\epsilon: MUZ_{(p)} \to BP$ [9, §3]. Since ϵ is multiplicative and $\epsilon[CP(p-1)] = v_1$, $\epsilon[V^{2p^2-2}] = v_2 \bmod v_1^{p+1}$ for any choice of the Milnor manifold $[V^{2p^2-2}]$, we see that

$$\epsilon(p^2, [CP(p-1)]^p, [V^{2p^2-2}]^p) = J.$$

Now if X is any finite complex,

(4) $BP^*(X) = MU^*(X) \otimes_{MU^*} BP^*$ [1, Proposition 25 and Note 9],

where BP^* becomes an MU^*-module via ϵ. Then Corollary 5 follows from Corollary 4.

Remarks. 1. Cobordism realizability and bordism realizability are equivalent by Spanier-Whitehead duality.

2. Actually, the ideal $(p^2, [CP(p-1)]^p, [V^{2p^2-2}]^p)$ is not invariant in MU^*. This follows from Proposition 1 and a result of Landweber [4, Theorem 6.2].

Proof of Theorem 1. If $J_t = (p^2, v_1^p, v_2^{tp}) \subset BP^*$, then

$$r_p v_2^{pt} = ptyv_1 v_2^{pt-1} \bmod J_t \neq 0 \bmod J_t$$

if $(t, p) = 1$. Hence J_t is not invariant, so M_t cannot be realized, proving the theorem.

Unfortunately, J_p is invariant, so the realizability of M_p remains an open question.

The gap. Now let p be odd, and let $q = 2p-2$. Toda's proof of (2) is indirect; he derives the equivalent result

$$p\pi^S_{p^2q-3} = 0.$$

We would like to prove (2) directly, using the following line of argument.

Proposition 6. *If $\zeta \in \pi^S_{p^2q-2}$ is not detected by any secondary BP operation, then $\zeta = 0$.*

The proof, which is not hard, is based on the BP secondary-operation theory in [10] and the edge-periodicity and vanishing theorems of [9, §§2 and 7].

Working in the stable category [2], we let $V'(1)$ be the four-cell complex constructed in [6, p. 249].

Proposition 7. *Any stable map $\Gamma: S^{p^2q-2}V'(1) \to S^0$ induces the zero map on BP-cohomology.*

Proof. If not,

$$BP^*(\Gamma)\colon BP^* \longrightarrow S^{(p^2+p)q}BP^*/(p^2, v_1^p)$$

must be the BP^*-module map taking the generator of the left-hand side to cv_2^p, $c \in Z_{(p)}$, $c \not\equiv 0 \mod p^2$. We may take $c = 1$, and it follows that if $C(\Gamma)$ is the mapping cone, $BP^*(C(\Gamma))$ is a BP^*-module of rank 2, with one generator (a nonzero element of maximal degree, hence primitive) having annihilator ideal (p^2, v_1^p, v_2^p). But this is impossible by Corollary 3. The proposition follows.

We can combine Propositions 6 and 7 using hopscotch diagrams like 3.2 and 3.3 of [10], to prove

Corollary 8. *The composition* $S^{p^2q-2} \hookrightarrow S^{p^2q-2}V'(1) \xrightarrow{\Gamma} S^0$ *is stably null-homotopic for any* Γ.

But to prove (2) we must cross the

Gap. Does any $\gamma\colon S^{p^2q-2} \longrightarrow S^0$ factor through $S^{p^2q-2}V'(1)$?

Toda-bracket manipulations show that this is at least plausible. Smith's proof of 4.7 in [6] can almost be reversed to derive (2) from (1), but again this Gap intervenes. A (presumably homotopy-theoretic) filling of the Gap would also provide a true intrinsic proof of the nontriviality of the first nonzero differential in the odd-primary BP Adams spectral sequance [9, §7].

REFERENCES

1. J. F. Adams, *Lectures on generalized cohomology*, Category Theory, Homology Theory and Their Applications, III (Battelle Inst. Conf., Seattle, Wash., 1968, vol. 3), Lecture Notes in Math., vol. 99, Springer-Verlag, Berlin, 1969, pp. 1–138. MR 40 #4943.

2. ———, *Stable homotopy and generalized homology*, Univ. of Chicago Lecture Notes, 1971.

3. M. Hazewinkel, *Constructing formal groups over $Z_{(p)}$-algebras*. I, Netherlands School of Economics (preprint).

4. P. S. Landweber, *Homological properties of comodules over $MU^*(MU)$ and $BP^*(BP)$*, Rutgers University, New Brunswick, N. J. (preprint).

5. A. Liulevicius, *On the algebra $BP^*(BP)$*, Lecture Notes in Math., vol. 249, Springer-Verlag, Berlin, 1972.

6. L. Smith, *On realizing complex bordism modules*. II. *Applications to the stable homotopy groups of spheres*, Amer. J. Math. 93 (1971), 226–263. MR 43 #1186b.

7. ———, *On realizing complex bordism modules*. III, Amer. J. Math. 94 (1972), 875–890. MR 46 #10014.

8. H. Toda, *An important relation in homotopy groups of spheres*, Proc. Japan Acad. 43 (1967), 839–842. MR 37 #5872.

9. R. Zahler, *The Adams-Novikov spectral sequence for the spheres*, Ann. of Math. (2) 96 (1972), 480–504.

10. ———, *Detecting stable homotopy with secondary cobordism operations*. I, Quart. J. Math. Oxford Ser. (2) 25 (1974), 213–226.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903