AN APPLICATION OF THE SEPARATION THEOREM FOR HERMITIAN MATRICES

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ABSTRACT. Suppose $H$ is an $n \times n$ hermitian matrix over the complex field partitioned as $H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$, where $C$ is invertible. Using the separation theorem on eigenvalues of hermitian matrices, bounds are obtained for the eigenvalues of $(H/C) = A - BC^{-1}B^*$ in terms of the eigenvalues of $H$ and $C$.

I. Introduction. Suppose $H$ is an hermitian matrix of order $n$ partitioned as

$$H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$ 

If $C$ is nonsingular, the Schur complement of $C$ in $H$ is $A - BC^{-1}B^* = (H/C)$. Haynsworth proved in [2] that the inertia of $H$, denoted $\text{In}(H)$, is $\text{In}(H/C) + \text{In}(C)$. The purpose of this paper is to determine bounds for the eigenvalues of $(H/C)$ in terms of the eigenvalues of $H$ and $C$. Our main tool will be the well-known interlacing theorem for hermitian matrices, which we now state for completeness.

Theorem [3]. Suppose $H$ is an $n \times n$ hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $A$ be the principal submatrix of $H$ obtained by deleting the $k$th row and $k$th column of $H$. If $\alpha_1 \geq \cdots \geq \alpha_{n-1}$ are the eigenvalues of $A$, then

$$\lambda_1 \geq \alpha_1 \geq \lambda_2 \geq \alpha_2 \geq \cdots \geq \alpha_{n-1} \geq \lambda_n.$$ 

From this classical theorem, it follows easily that if $A$ is a principal submatrix of $H$ of order $p$ with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_p$, then

$$\lambda_i \geq \alpha_i \geq \lambda_{n-p+i} \quad \text{for} \quad i = 1, \ldots, p.$$ 

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With regard to notation, we write $H(i, k, \ldots, n|j, k, \ldots, n)$ to denote the minor of $H$ with rows indexed by $(i, k, \ldots, n)$ and columns indexed by $(j, k, \ldots, n)$, where, of course, $1 \leq i, j \leq k - 1$. Also, sometimes we find it convenient to denote the eigenvalues of a $p \times p$ hermitian matrix, $M$, by $\lambda_1(M) \geq \cdots \geq \lambda_p(M)$.

II. Bounds for the eigenvalues of $(H/C)$. Assume $H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is hermitian of order $n$, $A$ is of order $k - 1$, and thus $C$ is of order $n - k + 1$. Further, suppose $C$ is invertible. Now, if we set $(H/C) = (d_{ij})$, then Crabtree and Haynsworth [1] have shown

$$d_{ij} = \frac{H(i, k, \ldots, n|j, k, \ldots, n)}{\det(C)} \quad \text{for } 1 \leq i, j \leq k - 1. \quad (3)$$

If we let $E = (e_{ij})$ where $e_{ij} = H(i, k, \ldots, n|j, k, \ldots, n)$ for $1 \leq i, j \leq k - 1$, then $(1/\det(C)) \cdot E = (H/C)$. It is easy to verify that $E$ is a principal submatrix of the $(n - k + 2)$-compound matrix of $H$, $C_{n-k+2}(H)$, which is hermitian. Then the eigenvalues of $C_{n-k+2}(H)$, say $\partial_1 \geq \cdots \geq \partial_n$, are the $\binom{n}{n-k+2}$ products $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{n-k+2}}$, where $1 \leq i_1 < i_2 < \cdots < i_{n-k+2} \leq n$ [4, p. 24], where each $\lambda_k$ is an eigenvalue of $H$.

Thus, using (2), we have

$$\partial_i \geq \lambda_i(E) \geq \partial_{n-k+2}^{\binom{n}{n-k+2}-(k-1)+i} \quad \text{for } i = 1, \ldots, k - 1. \quad (4)$$

Finally, if $\det(C) > 0$, we get

$$\frac{\partial_i}{\det(C)} \geq \frac{\lambda_i(H/C)}{\det(C)} \geq \partial_{n-k+2}^{\binom{n}{n-k+2}-(k-i)+1} / \det(C) \quad \text{for } i = 1, \ldots, k - 1. \quad (5)$$

We have proved

Theorem 1. Suppose $H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is an hermitian matrix with the dimensions of $A$ and $C$ as specified earlier. Assume $\det(C) > 0$, and let $C_{n-k+2}(H)$ denote the $(n - k + 2)$-compound matrix of $H$. If we denote the eigenvalues of $C_{n-k+2}(H)$, $C$, and $(H/C)$, respectively, by $\partial_1 \geq \cdots \geq \partial_{n-k+2}$,
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\[ \alpha_1 \geq \cdots \geq \alpha_{n-k+1}; \quad \text{and} \quad \beta_1 \geq \cdots \geq \beta_{k-1}, \text{ then} \]

\[ \frac{\partial_i}{\alpha_1 \cdots \alpha_{n-k+1}} \geq \beta_i \geq \frac{\left(\frac{n}{n-k+2}\right)^{-k+i+1}}{\alpha_1 \cdots \alpha_{n-k+1}} \quad \text{for } i = 1, \ldots, k-1. \]

Clearly, the above result holds a fortiori for \( H \) positive definite. In this case, \( \lambda_1 \cdots \lambda_{n-k+2} \) is the largest eigenvalue of \( C_{n-k+2}(H) \) and \( \lambda_{k-1} \cdots \lambda_n \) is the smallest eigenvalue of \( C_{n-k+2}(H) \), and we obtain a

Corollary. Under the hypotheses of the theorem with \( H \) positive definite, then

\[ \frac{\lambda_1 \cdots \lambda_{n-k+2}}{\alpha_1 \cdots \alpha_{n-k+1}} \geq \beta_i \geq \frac{\lambda_{k-1} \cdots \lambda_n}{\alpha_1 \cdots \alpha_{n-k+1}} \quad \text{for } i = 1, 2, \ldots, k-1. \]

We make two simple observations concerning the Corollary. For \( k = 2 \), the Corollary becomes

\[ \frac{\lambda_1 \cdots \lambda_n}{\alpha_1 \cdots \alpha_{n-1}} \geq \det(H/C) \geq \frac{\lambda_1 \cdots \lambda_n}{\alpha_1 \cdots \alpha_{n-1}}, \]

which yields \( \det(H) = \det(C)\det(H/C) \), a special case of Schur's identity [2, p. 74] since \( C \) is of order \( n-1 \). For \( k = 3 \), the Corollary yields

\[ \det(H/C)/\lambda_n \geq \beta_i \geq \det(H/C)/\lambda_1 \quad \text{for } i = 1, 2, \]

and thus \( 1/\lambda_n \geq 1/\beta_i \geq 1/\lambda_1 \) for \( i = 1, 2 \), a reciprocal separation property. Further, we obtain \( \lambda_1^2 \geq \beta_1 \beta_2 \geq \lambda_2^2 \) from the above inequality.

III. The positive definite case. Suppose \( A \) is an \( n \times n \) positive definite matrix. Denote by \( A_k \) the principal submatrix of \( A \) contained in rows \( 1, 2, \ldots, k \), for \( k = 1, \ldots, n-1 \), and let \( \lambda_n(A) \) be the minimal eigenvalue of \( A \). As before, \( \lambda_1(A) \) denotes the maximal eigenvalue of \( A \). The following theorem and proof is similar to a result of Watford [5, Theorem 4] on M-matrices.

Theorem 2. Suppose \( A \) is a positive definite matrix of order \( n \). Then

\[ \lambda_n(A) \leq \lambda_n(A|A_1) \leq \cdots \leq \lambda_n(A|A_{n-1}), \]

and

\[ \lambda_1(A|A_{n-1}) \leq \cdots \leq \lambda_1(A|A_1) \leq \lambda_1(A). \]

Proof. Assume, first, that \( A \) is partitioned as

\[ A = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix}. \]
Now $\lambda_1(A^{-1}) = 1/\lambda_n(A)$, and since $(A|B)^{-1}$ is a principal submatrix of $A^{-1}$ [5, p. 251], we have $\lambda_1[(A|B)^{-1}] \leq \lambda_1(A^{-1})$, using the separation theorem. Thus it follows that

$$\lambda_n(A) \leq \lambda_n(A|B).$$

Next, we note that if $B = A_{p+1}$ in (6), and

$$A_{p+1} = \begin{pmatrix}
A_p & B_{12} \\
B_{12}^* & a_{p+1,p+1}
\end{pmatrix};$$

then

$$(A|A_{p+1}) = ((A|A_p)|(A_{p+1}|A_p)),$$

the Haynsworth quotient property [1]. Using (7), we have

$$\lambda_n(A|A_{p+1}) \geq \lambda_n(A|A_p) \geq \lambda_n(A),$$

and statement (2.1) is immediate. We obtain (2.2) by noting $\lambda_n(A) = 1/\lambda_1(A)$.

IV. Conclusion. There exist matrices for which the bounds of Theorem 1 are exact. However, even if $H = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$, the bounds may give only rough estimates for the eigenvalues of $(H/C) = A$. For example, if $A = \begin{pmatrix} 3 \\ 1 \\ 3 \\ 4 \end{pmatrix}$ and $C = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$, then $H = A + C$ has eigenvalues $\lambda_1 = 7$, $\lambda_2 = 4$, $\lambda_3 = 2$, $\lambda_4 = 1$. But $(H/C) = A$ has eigenvalues $\beta_1 = 4$, $\beta_2 = 2$, and $C_{n-k+2}(H) = C_3(H)$ has eigenvalues $\delta_1 = 56$, $\delta_2 = 14$, $\delta_3 = 8$. The theorem yields $56/7 \geq \beta_1 \geq 14/7$ and $14/7 \geq \beta_2 \geq 8/7$. It seems likely that one could obtain "tighter" bounds in general by an application of the Courant minimax theorem for hermitian matrices—results which we shall not investigate in this paper.

REFERENCES


