FINITE GROUPS WHOSE SUBNORMAL SUBGROUPS PERMUTE WITH ALL SYLOW SUBGROUPS

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ABSTRACT. As a generalization of \((t)\)-groups and of \((q)\)-groups, a group \(G\) is called a \((\pi - q)\)-group if every subnormal subgroup of \(G\) permutes with all Sylow subgroups of \(G\). It is shown that if \(G\) is a finite solvable \((\pi - q)\)-group, then its hypercommutator subgroup \(D(G)\) is a Hall subgroup of odd order and every subgroup of \(D(G)\) is normal in \(G\); conversely, if a group \(G\) has a normal Hall subgroup \(N\) such that \(G/N\) is a \((\pi - q)\)-group and every subnormal subgroup of \(N\) is normal in \(G\), then \(G\) is a \((\pi - q)\)-group.

Let \(G\) be a finite group. Following Ore [6], a subgroup of \(G\) is called quasinormal in \(G\) if it permutes with all subgroups of \(G\). As a generalization of this concept we say, following Kegel, that a subgroup of \(G\) is \(\pi\)-quasinormal in \(G\) if it permutes with all Sylow subgroups of \(G\). Kegel [5] proved that a \(\pi\)-quasinormal subgroup is always a subnormal subgroup. Since a subnormal subgroup is not necessarily a \(\pi\)-quasinormal subgroup, it seems natural to ask: "What can be said about the structure of a group if all of its subnormal subgroups are \(\pi\)-quasinormal in the group?" We call such a group a \((\pi - q)\)-group and, in this paper, we study finite \((\pi - q)\)-groups. We especially study and characterize, in \(\S\) 2, finite solvable \((\pi - q)\)-groups. Theorems 2.3, 2.4, and 2.5 establish the characterization.

The \((q)\)-groups and the \((t)\)-groups (see the definitions below) have been studied by Zacher [8] and Gaschütz [2], respectively, and they have characterized such finite solvable groups. We extend their results to finite solvable \((\pi - q)\)-groups and give the conditions under which a \((\pi - q)\)-group is either a \((q)\)-group or a \((t)\)-group.

Throughout the paper, the groups are finite.

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1. Definitions and preliminary results.

Definitions. Subgroups $H$ and $K$ of the group $G$ permute if $HK = KH$. A subgroup of $G$ is \(\pi\)-quasinormal (quasinormal) in $G$ if it permutes with all Sylow subgroups (all subgroups) of $G$. A group $G$ is a \((\pi - q)\)-group ((\(q)\)-group) if every subnormal subgroup of $G$ is \(\pi\)-quasinormal (quasinormal) in $G$, and $G$ is a \((\pi)\)-group if all subnormal subgroups of $G$ are normal in $G$.

We now mention two basic results on \(\pi\)-quasinormal subgroups which are needed later.

(1.1) [5] If $H < K < G$ and $H$ is \(\pi\)-quasinormal in $G$, then $H$ is \(\pi\)-quasinormal in $K$.

(1.2) [5] If $\alpha: G \rightarrow G^\alpha$ is a homomorphism from $G$ onto $G$ and $H$ is a \(\pi\)-quasinormal subgroup of $G$, then $H^\alpha$ is \(\pi\)-quasinormal in $G^\alpha$.

Remark. In view of (1.1), a \((\pi - q)\)-group can be defined as a group in which \(\pi\)-quasinormality is a transitive relation.

The following two inheritance properties of \((\pi - q)\)-groups are immediate consequences of (1.1) and (1.2).

(1.3) A subnormal subgroup of a \((\pi - q)\)-group is again a \((\pi - q)\)-group. But a nonsubnormal subgroup of a \((\pi - q)\)-group is not necessarily a \((\pi - q)\)-group.

(1.4) A factor group of a \((\pi - q)\)-group is a \((\pi - q)\)-group.

Proposition 1.5. If $G_1$ and $G_2$ are two \((\pi - q)\)-groups and $(|G_1|, |G_2|) = 1$, then $G = G_1 \times G_2$ is also a \((\pi - q)\)-group.

Proof. Let $H$ be a subnormal subgroup of $G = G_1 \times G_2$ and $G_p$ be a Sylow $p$-subgroup of $G$. To prove that $G$ is a \((\pi - q)\)-group, we must show that $H$ and $G_p$ permute. Since $H$ is subnormal in $G$ and $(|G_1|, |G_2|) = 1$, it is easily verified that $H = (H \cap G_1) \times (H \cap G_2)$. Clearly, we may assume without loss of generality that $G_p \leq G_1$. Since $H \cap G_1$ is subnormal in $G_1$ and $G_1$ is a \((\pi - q)\)-group, we see that $H \cap G_1$ permutes with $G_p$. Moreover, $H \cap G_2$ centralizes $G_p$. Hence $(H \cap G_1) \times (H \cap G_2) = H$ permutes with $G_p$. This proves the proposition.

Remark. In this proposition, the condition that $(|G_1|, |G_2|) = 1$ is necessary. The following example shows this.

Let $G_1 = S_3 = \langle x, y | x^3 = y^2 = 1, yx = x^2y \rangle$ and $G_2 = \langle z | z^3 = 1 \rangle$. Then $\langle xz \rangle$ is subnormal in $G_1 \times G_2$. But $\langle xz \rangle$ is not \(\pi\)-quasinormal in $G_1 \times G_2$ since it does not permute with the Sylow 2-subgroup $\langle y \rangle$ of $G_1 \times G_2$.

Remark. Note that a nilpotent group is always a \((\pi - q)\)-group but not
necessarily a \((q)\)-group and that the class of \((q)\)-groups defined by Zacher [8] is properly contained in the class of \((\pi - q)\)-groups.

2. Solvable \((\pi - q)\)-groups. In this section we will characterize the solvable \((\pi - q)\)-groups. We begin with the following observation.

**Lemma 2.1.** Let \(G\) be a \((\pi - q)\)-group. If \(N\) is a solvable minimal normal subgroup of \(G\), then the order of \(N\) is a prime.

**Proof.** Since \(N\) is a solvable minimal normal subgroup of \(G\), \(|N| = p^n\) for some prime \(p\). Hence every subgroup of \(N\), being subnormal in \(G\), is \(\pi\)-quasinormal in \(G\). Let \(G_p\) be any Sylow \(p\)-subgroup of \(G\). Then \(N\) is a normal subgroup of \(G_p\) and \(N \cap Z(G_p) \neq 1\), where \(Z(G_p)\) is the center of \(G_p\). Let \(g\) be a nonidentity element of \(N \cap Z(G_p)\). Since \(\langle g \rangle\) is \(\pi\)-quasinormal in \(G\), it follows that \(\langle g \rangle\) is subnormal in \(\langle g \rangle G = G_q\langle g \rangle\) for all Sylow \(q\)-subgroups \(G_q\) of \(G\) for primes \(q \neq p\). From this and the fact that \(\langle g \rangle\) is a Sylow \(p\)-subgroup of \(\langle g \rangle G_q\), we obtain that \(\langle g \rangle \lhd \langle g \rangle G_q\) for all \(G_q, q \neq p\). But \(\langle g \rangle \lhd G_p\), and hence \(\langle g \rangle \lhd G\). Since \(N\) is a minimal normal subgroup of \(G\), we have \(N = \langle g \rangle\) and so \(|N| = p\), as desired.

**Theorem 2.2.** A solvable \((\pi - q)\)-group \(G\) is supersolvable.

**Proof.** We use induction on the order of \(G\). Let \(N\) be a minimal normal subgroup of \(G\). Since \(G/N\) is a solvable \((\pi - q)\)-group, \(G/N\) is supersolvable by induction. But \(|N|\) is a prime by Lemma 2.1 and so \(G\) is supersolvable.

**Theorem 2.3.** Let \(G\) be a solvable \((\pi - q)\)-group and \(D(G)\) be its hypercommutator subgroup (the smallest normal subgroup of \(G\) such that \(G/D(G)\) is nilpotent). Then

(i) \(D(G)\) is a Hall subgroup of odd order, and

(ii) every subgroup of \(D(G)\) is normal in \(G\).

In particular, \(D(G)\) is an abelian subgroup of \(G\).

**Remark.** Note that every complement of \(D(G)\) in \(G\) is nilpotent.

**Proof.** We proceed by induction on the order of \(G\). Let \(p\) be the largest prime dividing \(|G|\) and \(G_p\) a Sylow \(p\)-subgroup of \(G\). Then, since \(G\) is supersolvable by Theorem 2.2, the order of \(D(G)\) is odd and \(G_p\) is normal in \(G\). We now have two cases according to whether or not \(p\) divides \(|D(G)|\).
Case 1. If $p$ does not divide $|D(G)|$, then $G_p$ centralizes $D(G)$. By induction, $D(G/G_p) = D(G)G_p/G_p$ is a Hall subgroup of $G/G_p$ and every subgroup of $D(G)G_p/G_p$ is normal in $G/G_p$. This means that $D(G)$ is a Hall subgroup of $G$ and, for $H \leq D(G)$, $HG_p$ is normal in $G$. But $H$ is a normal Hall (hence characteristic) subgroup of $HC_p$. Thus $H \triangleleft G$.

Case 2. If $p$ divides $|D(G)|$, then we will show that $G_p \leq D(G)$. Let $a \in G_p$. Since $\langle a \rangle$ is subnormal in $G$, $\langle a \rangle$ is $\pi$-quasinormal in $G$. Let $K$ be a $p$-complement of $G$. Then $\langle a \rangle K$ is a subgroup. Since $\langle a \rangle$ is a subnormal Sylow $p$-subgroup of $\langle a \rangle K$, $\langle a \rangle$ is normal in $\langle a \rangle K$, i.e., $K$ normalizes every subgroup of $G_p$. Hence every element of $K$ induces a power automorphism in the elementary abelian group $G_p/\Phi(G_p)$ of order prime to $p$, where $\Phi(G_p)$ is the Frattini subgroup of $G_p$. Thus, for every $k \in K$, there exists a positive integer $m(k)$ such that $a^k = a^m(k) \mod \Phi(G_p)$ for all $a \in G_p$. Let $\Gamma_n(G)$ be the terminal member of the lower central series of $G$. Then $\Gamma_n(G) = D(G)$. Since $p$ divides $|D(G)|$, $K$ does not centralize $G_p$. Hence there is some $y$ in $K$ which does not centralize $G_p$. Now it follows from [7, Theorem 11.7] that $m(y) \equiv 1 \mod p$ and so, for all $x \in G_p - \Phi(G_p)$, the commutator $((\ldots((x, y_1), y_2),\ldots), y_n) = x^{(m(y)-1)n} \equiv 1 \mod \Phi(G_p)$, where $y_1 = y_2 = \cdots = y_n = y$. This says that if $A/\Phi(G_p) \leq G_p/\Phi(G_p)$ and $|A/\Phi(G_p)| = p$, then $\Gamma_n(G)$ contains an element $g$ such that $g \in G_p - \Phi(G_p)$ and $g\Phi(G_p)$ generates $A/\Phi(G_p)$. The Burnside basis theorem yields that $G_p \leq \Gamma_n(G) = D(G)$. By induction, $D(G/G_p) = D(G)G_p/G_p = D(G)/G_p$ is a Hall subgroup of $G/G_p$, which implies that $D(G)$ is a Hall subgroup of $G$.

Next we prove that every subgroup of $D(G)$ is normal in $G$. Clearly, we need only show that every cyclic subgroup of $D(G)$ is normal in $G$. Let $\langle c \rangle$ be any cyclic subgroup of $D(G)$. Then $c = uv = vu$ for some $p$-element $u$ and $p'$-element $v$ and $\langle c \rangle = \langle u \rangle \times \langle v \rangle$. Since $G$ is supersolvable, $G'$ is nilpotent. But $D(G) \leq G'$ and so $D(G)$ is nilpotent. Hence all subgroups of $D(G)$, in particular $\langle u \rangle$ and $\langle v \rangle$, are $\pi$-quasinormal in $G$. It now follows that $\langle u \rangle \triangleleft \langle u \rangle G_q = G_q \langle u \rangle$ for all Sylow $q$-subgroups $G_q$ and primes $q \neq p$. Therefore, $\langle u \rangle \triangleleft G_p$, the normal subgroup of $G$ generated by all $p'$-elements of $G$. But $D(G) \leq G[p]$ and $G_p \leq D(G)$. Hence $G[p] = G$, and so $\langle u \rangle \triangleleft G$. Since $\langle v \rangle G_p/G_p$ is a subgroup of $D(G)/G_p = D(G/G_p)$, we have, by induction, that $\langle v \rangle G_p/G_p$ is normal in $G/G_p$. Thus $\langle v \rangle G_p \triangleleft G$.

But $\langle v \rangle G_p$ is nilpotent. Hence $\langle v \rangle$ is characteristic in $\langle v \rangle G_p$ and so $\langle v \rangle \triangleleft G$. Therefore, $\langle c \rangle \triangleleft G$ and this takes care of Case 2.
Finally, since $D(G)$ is Hamiltonian of odd order, it follows that $D(G)$ is abelian. Hence the proof of the theorem is complete.

The next theorem gives sufficient conditions for a group $G$ to be a $(\pi - q)$-group. Here we do not require that $G$ be solvable.

**Theorem 2.4.** Let the group $G$ have a normal Hall subgroup $N$ such that:

(i) $G/N$ is a $(\pi - q)$-group, and

(ii) every subnormal subgroup of $N$ is normal in $G$.

Then $G$ is a $(\pi - q)$-group.

**Proof.** Let $H$ be a subnormal subgroup of $G$. Then we must show that $H$ is $\pi$-quasinormal in $G$. Let $N \vartriangleleft H$. Since $N \cap H \neq 1$. Since $N \cap H$ is subnormal in $N$, $N \cap H$ is normal in $G$ by (ii). Now consider $G/N \cap H$. By induction, $H/N \cap H$ is $\pi$-quasinormal in $G/N \cap H$. From this it follows that $H$ is $\pi$-quasinormal in $G$.

Next suppose that $N \cap H = 1$. By the improved version of the Schur-Zassenhaus theorem (after the well-known theorem of Feit and Thompson), $G$ splits over $N$ and all complements of $N$ are conjugate. Let $M$ be any complement of $N$ in $G$. Then $M$, being isomorphic to $G/N$, is a $(\pi - q)$-group. Since $H$ is subnormal in $G$ and $(|N|, |M|) = 1$, it is easily checked that $H = (H \cap M)(H \cap N)$. But $H \cap N = 1$ and so $H = H \cap M$. This means that $H \leq M$. Hence every complement of $N$ is a $(\pi - q)$-group and contains $H$. Note that $(|H|, |N|) = 1$.

Now consider the subgroup $HN$. Since $H$ is subnormal in $HN$ and $(|H|, |N|) = 1$, it follows that $H$ is characteristic in $HN$. Hence $H$ permutes with every Sylow subgroup of $N$. Let $p$ be a prime divisor of the order of $G$ and $G_p$ be a Sylow $p$-subgroup of $G$. If $p$ divides the order of $N$, then $G_p \leq N$ and so $HG_p = G_p H$. On the other hand, if $p$ does not divide $|N|$, then there exists a complement $L$ of $N$ in $G$ such that $G_p \leq L$. Since $H$ is a subnormal subgroup of $L$ and $L$ is a $(\pi - q)$-group, the subgroups $H$ and $G_p$ permute. Hence $H$ is $\pi$-quasinormal in $G$. This proves the theorem.

From this we obtain the following result and see that conditions (i) and (ii) of Theorem 2.3 are not only necessary but are also sufficient.

**Theorem 2.5.** Let $G$ have a normal Hall subgroup $N$ such that:

(i) $G/N$ is a solvable $(\pi - q)$-group, and

(ii) $N$ is solvable and all its subnormal subgroups are normal in $G$.

Then $G$ is a solvable $(\pi - q)$-group.
Proof. Since $G/N$ and $N$ are solvable, $G$ is solvable. The rest follows from Theorem 2.4.

Remark. Condition (i) of Theorem 2.5 is automatically satisfied if the factor group is nilpotent.

Corollary 2.6. Let $G$ be a solvable $(\pi - q)$-group. Then its subgroups are again solvable $(\pi - q)$-groups.

Proof. Let $K$ be a subgroup of $G$ and consider $K \cap D(G)$. It follows from Theorem 2.3 that $K \cap D(G)$ is a normal Hall subgroup of $K$ and its subnormal subgroups are normal in $K$. Also, $D(G)K/D(G) \cong K/K \cap D(G)$ and so $K/K \cap D(G)$ is nilpotent. Now $K$ is a solvable $(\pi - q)$-group by Theorem 2.5.

3. $(\pi - q)$-groups with special Sylow subgroups. In this section we show when a $(\pi - q)$-group, not necessarily solvable, is a $(q)$-group or a $(t)$-group. We need the following definition.

Definition. A group $G$ is called quasi-Hamiltonian if all of its subgroups are quasinormal in $G$.

Iwasawa [4] has shown the existence of quasi-Hamiltonian $p$-groups that are not Hamiltonian. This suggests the next theorem.

Theorem 3.1. Let $G$ be a $(\pi - q)$-group. If all Sylow subgroups of $G$ are quasi-Hamiltonian, then $G$ is a $(q)$-group.

Proof. Let $K$ be a subnormal subgroup of $G$. Then we must show that $K$ is quasinormal in $G$. Since the factor groups of $G$ satisfy the conditions of the theorem, it is sufficient to consider the case when the core of $K$ in $G$ (the largest normal subgroup of $G$ contained in $K$) is $\{1\}$. Note that $K$ is certainly $\pi$-quasinormal in $G$. By Deskins [1], $K$ is nilpotent. Hence every subgroup of $K$ is $\pi$-quasinormal in $G$. Let $p$ be a prime divisor of $|K|$. Then the Sylow $p$-subgroup $K_p$ of $K$ is $\pi$-quasinormal in $G$. Therefore, $K_p$ is normalized by every $p'$-element of $G$ and it is contained in every Sylow $p$-subgroup of $G$. But Sylow subgroups of $G$ are quasi-Hamiltonian. Hence $K_p$ permutes with every $p$-subgroup of $G$. Now let $g$ be any element of $G$. Then $\langle g \rangle = \langle u \rangle \times \langle v \rangle$ for some $p$-element $u$ and $p'$-element $v$, and so $\langle g \rangle$ and $K_p$ permute. Since $K$ is the direct product of its Sylow subgroups, it follows that $K$ and $\langle g \rangle$ permute, which implies that $K$ is quasinormal in $G$. This completes the proof.

Theorem 3.2. Let $G$ be a $(\pi - q)$-group. If all Sylow subgroups of $G$ are Hamiltonian, then $G$ is a $(t)$-group.
Proof. Let $K$ be subnormal in $G$. Then, as above, $K_p$ is normalized by every $p'$-element of $G$ and, since Sylow subgroups of $G$ are Hamiltonian, $K_p$ is a normal subgroup of every Sylow $p$-subgroup of $G$. Hence $K_p \triangleleft G$, and so $K \triangleleft G$.

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