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NEARLY COMONOTONE APPROXIMATION

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ABSTRACT. This paper obtains estimates on the degree of nearly comonotone approximation which extend and improve the estimate obtained by Newman, Passow, and Raymon. In particular, the restriction that $f \in \mathrm{Lip}_M 1$ is removed, and estimates for the degree of nearly comonotone approximation are obtained for all proper piecewise monotone functions. It is also shown that if f' exists and is continuous on the interval, then the ordinary polynomials of best approximation form a nearly comonotone sequence.

1. **Introduction.** Let f be a continuous piecewise monotone function on $[a, b]$. By the *peaks* of f we mean the endpoints a, b and the relative maxima and minima. We number them from left to right $a = x_0 < x_1 < \cdots < x_m = b$. Thus f is monotone on each subinterval of the form $[x_{i-1}, x_i]$, $i = 1, 2, \cdots, m$. The monotonicity changes at each x_i. The problem studied here is the degree of approximation of f by polynomials which are monotone on each interval $[x_{i-1}, x_i]$ in the same sense as f. This type of problem is studied by Newman, Passow, and Raymon in [4]. This is a generalization of monotone approximation as studied by Shisha [7], Roulier [5], [6], Lorentz [1], and Lorentz and Zeller [2], [3]. In the case of monotone approximation the peaks are the endpoints a and b.

In the paper of Newman, Passow, and Raymon [4], slightly stronger assumptions on the function f are made. They define a *proper piecewise monotone function* f as follows:

For each $\epsilon > 0$ there is a $\delta > 0$ so that $|(f(x) - f(y))/(x - y)| \geq \delta$ for $x \neq y$ and x and y in $[x_{i-1} + \epsilon, x_i - \epsilon]$, $i = 1, \cdots, m$. (We assume $\epsilon < \frac{1}{2}\min_i(x_i - x_{i-1})$.) They also define the notions of comonotone and nearly comonotone sequences of polynomials. As above we assume the peaks of f are $x_0 < x_1 < \cdots < x_m$.

Definition. A sequence of algebraic polynomials $\{P_n\}$ is said to be *comonotone with* f on $[a, b]$, if for each n, P_n has the same monotonicity as f on $[x_{i-1}, x_i]$ for $i = 1, \cdots, m$. $\{P_n\}$ is said to be *nearly comonotone*

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with f on $[a, b]$, if for every $\epsilon > 0$ which satisfies $\epsilon < \frac{1}{2}\min_i(x_i - x_{i-1})$, and for n sufficiently large, P_n has the same monotonicity as f on $[x_{i-1} + \epsilon, x_i - \epsilon]$ for $i = 1, \cdots, m$. Let π_n be the algebraic polynomials of degree n or less. In [4] the main theorem is the following:

Theorem A (Newman, Passow, Raymon). *If* $f \in \lim_M 1$ on $[a, b]$ *and a proper piecewise monotone function on* $[a, b]$, *then there is a nearly comonotone sequence* $\{P_n\}$, $P_n \in \pi_n$, *such that*

$$\max_{a\le x\le b} |f(x) - P_n(x)| \le CM/n.$$

In this paper we use the techniques in Roulier [5], [6] for a simpler approach to obtain new results. That is, we obtain an estimate for the degree of nearly comonotone approximation without assuming f is $\mathrm{Lip}_M 1$, and we show that if f' is continuous then the polynomials of best approximation for f are a nearly comonotone sequence.

2. **The main theorem.** *If* f *is continuous on* $[a, b]$ *and* n *a nonnegative integer, we define*

$$E_n(f) = \inf_{P_n \in \pi_n} \max_{a\le x\le b} |f(x) - P_n(x)|.$$

That is, $E_n(f)$ *is the degree of best approximation to* f *by polynomials of degree* n *or less.*

In the following theorem we let $E_n = E_n(f)$, and use the technique used in [5].

Theorem 1. *Let* f *be a proper piecewise monotone function on* $[a, b]$ *and let* w *be its modulus of continuity. Let* $\{D_n\}_{n=0}^{\infty}$ *be a sequence of positive numbers satisfying* $D_n \to 0$ *as* $n \to +\infty$, *and* $E_n/D_n \to 0$ *as* $n \to +\infty$. *Then there is a nearly comonotone sequence* $\{P_n\}_{n=0}^{\infty}$ *such that for each* n

$$\max_{a\le x\le b} |f(x) - P_r(x)| \le w(D_n) + E_n.$$

(*Without loss of generality, we assume* $D_n \le b - a$, $n = 0, 1, \cdots$.)

Proof. We assume as above that the peaks of f are at $a = x_0 < x_1 < \cdots < x_m = b$.

Let p_n be the polynomial of best approximation to f from π_n on $[a, b]$. Define

$$\alpha_n(x) = (1 - D_n/(b - a))(x - a) + a \quad \text{and} \quad \beta_n(x) = \alpha_n(x) + D_n.$$

Let

(1) $$P_n(x) = \frac{1}{D_n}\int_{\alpha_n(x)}^{\beta_n(x)} p_n(t)\,dt.$$

Then

(2) $$P_n'(x) = \frac{1}{D_n}\,[p_n(\beta_n(x)) - p_n(\alpha_n(x))]\cdot\left(1 - \frac{D_n}{b-a}\right).$$

Observe

(3) $$p_n(x) - p_n(y) = p_n(x) - f(x) + f(x) - f(y) + f(y) - p_n(y).$$

Thus

(4) $$f(x) - f(y) - 2E_n \le p_n(x) - p_n(y) \le f(x) - f(y) + 2E_n.$$

Let ϵ be given so that $0 < \epsilon < \frac{1}{4}\min_i(x_i - x_{i-1})$.

Let $\delta > 0$ correspond to ϵ as in the definition of proper piecewise monotone. That is

(5) $$\left|\frac{f(x) - f(y)}{x - y}\right| \ge \delta$$

for $x \ne y$, $x, y \in [x_{i-1} + \epsilon, x_i - \epsilon]$, $i = 1, \cdots, m$. Thus by (4) and (5) it follows that on an interval $[x_{i-1} + \epsilon, x_i - \epsilon]$ where f is increasing, we have for $x > y$ in this interval,

(6) $$p_n(x) - p_n(y) \ge \delta(x - y) - 2E_n.$$

Similarly, on an interval $[x_{j-1} + \epsilon, x_j - \epsilon]$ where f is decreasing we have for $x > y$ in this interval

(7) $$p_n(x) - p_n(y) \le 2E_n - \delta(x - y).$$

We also see easily that for all x in $[a, b]$ and $n = 0, 1, \cdots$, we have

(8) $$a \le \alpha_n(x) \le x \le \beta_n(x) \le b,$$

and

(9) $$0 \le x - \alpha_n \le D_n,$$

and

(10) $$0 \le \beta_n - x \le D_n.$$

Choose N for which $n \ge N$ implies $D_n < \epsilon$.

If we consider $[x_{i-1} + 2\epsilon, x_i - 2\epsilon]$ where f is increasing, we see that for $n \geq N$ and $x_{i-1} + 2\epsilon \leq x \leq x_i - 2\epsilon$, we have

$$x_{i-1} + \epsilon \leq \alpha_n(x) < \beta_n(x) \leq x_i - \epsilon. \tag{11}$$

Thus, using (2) and (6) and (11)

$$\begin{aligned} P_n'(x) &= \frac{1}{D_n}\left(1 - \frac{D_n}{b-a}\right)[p_n(\beta_n(x)) - p_n(\alpha_n(x))] \\ &\geq \frac{1}{D_n}\left(1 - \frac{D_n}{b-a}\right)[\delta(\beta_n(x) - \alpha_n(x)) - 2E_n] \qquad (12) \\ &\geq \frac{1}{D_n}\left(1 - \frac{D_n}{b-a}\right)[\delta D_n - 2E_n] = \left(1 - \frac{D_n}{b-a}\right)[\delta - 2E_n/D_n]. \end{aligned}$$

Similarly, if f is decreasing on $[x_{j-1} + 2\epsilon, x_j - 2\epsilon]$ we see that for $n \geq N$ and $x_{j-1} + 2\epsilon \leq x \leq x_j - 2\epsilon$ we have using (7)

$$P_n'(x) \leq (1 - D_n/(b-a))(2E_n/D_n - \delta). \tag{13}$$

By (12) and (13) if we choose n so large that $2(E_n/D_n) < \delta$ then f and P_n agree in monotonicity on $[x_{i-1} + 2\epsilon, x_i - 2\epsilon]$ for $i = 1, 2, \cdots, m$. Thus $\{P_n\}$ is a nearly comonotone sequence.

To prove the degree of approximation, observe that for $a \leq x \leq b$ and $n = 0, 1, 2, \cdots$, we use (8), (9), and (10) to observe that

$$\begin{aligned} |f(x) - P_n(x)| &= \left| \frac{1}{D_n}\int_{\alpha_n(x)}^{\beta_n(x)} f(x)\,dt - \frac{1}{D_n}\int_{\alpha_n(x)}^{\beta_n(x)} p_n(t)\,dt \right| \\ &\leq \frac{1}{D_n}\int_{\alpha_n(x)}^{\beta_n(x)} |f(x) - f(t)|\,dt + \frac{1}{D_n}\int_{\alpha_n(x)}^{\beta_n(x)} |f(t) - p_n(t)|\,dt \\ &\leq w(D_n) + E_n. \quad \square \end{aligned}$$

Roulier [6] proves the following

Lemma. *Let $f \in C^k[a, b]$. Suppose that $a < a' < b' < b$. If for a sequence of algebraic polynomials $\{P_n\}$ (P_n of degree n or less) the condition*

$$\max_{a \leq x \leq b} |f(x) - P_n(x)| = o(n^{-k})$$

is satisfied then

$$\max_{a' \leq x \leq b'} |f^{(j)}(x) - P_n^{(j)}(x)| = o(n^{j-k}), \qquad j = 1, 2, \cdots, k.$$

This gives the following

Theorem 2. *Let f be a proper piecewise monotone function on $[a, b]$ and assume f' is continuous on $[a, b]$. For each $n = 0, 1, \cdots$ let P_n be the polynomial of best approximation to f from π_n on $[a, b]$. Then $\{P_n\}$ is a nearly comonotone sequence for f.*

Proof. Observe that by a theorem due to Jackson, $E_n(f) = o(1/n)$. Thus, by the lemma $P_n' \to f'$ on any interval $[a', b']$ with $a < a' < b' < b$. □

3. **Remarks.** Theorem 1 shows that if we assume $f \in \mathrm{Lip}_M 1$ then the degree of nearly comonotone approximation is almost as good as the degree of ordinary approximation for f. Theorem 2 shows that for $f' \in C[a, b]$ they are the same.

Theorems are also possible showing an estimate of the degree of comonotone approximation on the whole interval for certain cases. But understandably they are not as good or as general as those above. These will appear elsewhere.

Theorem 2 cannot be improved by replacing the assumption that f is proper piecewise monotone with piecewise monotone.

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