PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 47, Number 1, January 1975

ON THE EXISTENCE OF κ-FREE ABELIAN GROUPS

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ABSTRACT. It is proved that if $\aleph_\alpha$ is a regular cardinal such that there is an $\aleph_\alpha$-free abelian group which is not $\aleph_{\alpha+1}$-free, then for every positive integer n there is an $\aleph_{\alpha+n}$-free abelian group which is not $\aleph_{\alpha+n+1}$-free. A corollary is that for each positive integer n there is a group of cardinality $\aleph_n$ which is $\aleph_n$-free but not free. Some results on κ-free abelian groups which involve notions from logic are also proved.

Introduction. Throughout this paper the word "group" will mean abelian group. If A is a group and κ is an infinite cardinal, A is said to be *κ-free* if every subgroup of A generated by fewer then κ elements is free. Thus A is $\aleph_0$-free if and only if A is torsion-free; and for an uncountable κ, A is κ-free if and only if every subgroup of A of a cardinality $<\kappa$ is free. Problem 10 of [4] asks for which κ are there groups which are κ-free but not κ^+-free. In this paper, we prove that if $\kappa = \aleph_\alpha$ has this property and κ is regular, then so does $\aleph_{\alpha+n}$ for each $n < \aleph_0$ (Corollary 2.4). As a consequence, we obtain the result that for each $n < \aleph_0$ there is an $\aleph_n$-free group which is not $\aleph_{n+1}$-free. This improves a result of Griffith [7] which proved the existence of nonfree $\aleph_n$-free groups for each n but did not determine the cardinality of such groups.

In § 3, we state some other consequences of our techniques which involve concepts from logic. We strengthen the main theorem of § 2 to prove that if $\aleph_\alpha$ is regular and there is an $\aleph_\alpha$-free group which is not $\aleph_{\alpha+1}$-free, then for any $1 \le n < \omega$ if $\kappa = \aleph_{\alpha+n}$ there is a group of cardinality κ which is $L_{\infty\kappa}$ equivalent to a free group but which is not free (Theorem 3.1). As a corollary we also obtain a proof of a conjecture of Hill (Corollary 3.2). We also indicate how our techniques can be used to prove a theorem of Gregory on a consequence of the axiom of constructibility (Theorem 3.5).

After obtaining our results, we learned that Hill had independently found proofs of Corollaries 2.3 and 3.2. (Earlier he had proved the existence

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Received by the editors November 26, 1973.

AMS (MOS) subject classifications (1970). Primary 20K20; Secondary 13L05.

Key words and phrases. κ-free abelian group, stationary set, equivalence in $L_{\infty\kappa}$, axiom of constructibility, compact cardinal.

[1] Research partially supported by NSF Grant GP34091X.

of an $\aleph_2$-free group which is not $\aleph_3$-free [10].) Our proofs are quite different from his [12]. Interestingly, our proof is based on a construction which was first used by Hill to prove the existence of a nontrivial Fuchs 5 group [9] and then adapted by Griffith to construct a nonfree $\aleph_1$-separable group [7]. Our extension of this construction technique involves the use of a notion from set theory, that of a stationary set, which we discuss in § 1.

Finally, we remark that our results may all be translated into results about p-groups such that every subgroup of cardinality $<\kappa$ is a direct sum of cyclic groups. Thus, we give a partial solution to problem 56 of [5].

1. **Stationary sets.** We shall always identify an ordinal with the set of its predecessors: $\mu = \{\nu: \nu < \mu\}$; thus $C \subseteq \mu$ means C is a set of ordinals which are less than μ. If ν is an ordinal, $\nu + 1$ denotes the immediate successor of ν. An ordinal of the form $\nu + 1$ is called a *successor ordinal*; otherwise, it is a *limit ordinal*. A *cardinal* is an ordinal which cannot be put in one-one correspondence with any of its predecessors. If κ is a cardinal, κ^+ denotes the first cardinal which is strictly larger than κ. We define $\aleph_\alpha$ by induction on ordinals α: $\aleph_0 =$ the first infinite cardinal; $\aleph_\alpha = \sup\{\aleph_\beta^+: \beta < \alpha\}$.

If λ is a limit ordinal, the *cofinality* of λ is the smallest ordinal κ such that there is a strictly-increasing function $f: \kappa \to \lambda$ such that $\sup\{f(\nu): \nu < \kappa\} = \lambda$. We write $\mathrm{cf}(\lambda) = \kappa$; it may be proved that κ is an infinite cardinal $\leq \lambda$. In fact, $\mathrm{cf}(\lambda)$ is regular, where a cardinal κ is said to be *regular* if $\mathrm{cf}(\kappa) = \kappa$. Any cardinal of the form κ^+ is regular.

Let λ be a limit ordinal. A subset $C \subseteq \lambda$ is *closed in* λ if it is closed in the order topology i.e. for any $S \subseteq C$, $\sup S \in C$ whenever $\sup S \in \lambda$. $C \subseteq \lambda$ is *unbounded in* λ if $\sup C = \lambda$. $E \subseteq \lambda$ is *stationary in* λ if for any closed unbounded $C \subseteq \lambda$, $C \cap E \neq \emptyset$. Any closed unbounded subset of λ is stationary in λ. Another example—and a very useful one for our purposes—is provided by the following lemma. (We thank R. Solovay for informing us of this result.) Hereafter λ denotes a limit ordinal.

1.1 **Lemma.** *Let κ be a regular cardinal and let $E = \{\lambda < \kappa^+: \mathrm{cf}(\lambda) = \kappa\}$. Then E is stationary in κ^+. Moreover, E has the property that for any limit ordinal $\mu < \kappa^+$, $E \cap \mu$ is not stationary in μ.*

Proof. If C is a closed unbounded subset of κ^+, let $\{c_\nu: \nu < \kappa\}$ be any strictly increasing sequence in C. (Such a sequence exists because C is unbounded and $\mathrm{cf}(\kappa^+) = \kappa^+$.) Let $\lambda = \sup\{c_\nu: \nu < \kappa\}$. Since κ^+ is regular, $\lambda < \kappa^+$. Since C is closed, $\lambda \in C$. But clearly $\mathrm{cf}(\lambda) = \kappa$, since κ is regular, so $\lambda \in C \cap E$. Thus, E is stationary in κ^+. If μ is a limit ordinal

$<\kappa^+$, there is a strictly increasing sequence $S = \{\rho_\nu : \nu < \mathrm{cf}(\mu)\} \subseteq \mu$ whose limit is μ. We may assume S is closed in μ. If ρ_σ is a limit point of S, then $\mathrm{cf}(\rho_\sigma) < \kappa$, since ρ_ν is the limit of the sequence $\{\rho_\nu : \nu < \sigma\}$ which has cardinality $\leq \sigma < \mathrm{cf}(\mu) \leq \kappa$. Therefore, if we let $C = \{\rho_\nu + 1 : \rho_\nu$ is not a limit point of $S\} \cup \{\rho_\nu : \rho_\nu$ is a limit point of $S\}$, then C is a closed unbounded subset of μ such that $C \cap E = \emptyset$. Hence $E \cap \mu$ is not stationary in μ. This completes the proof of Lemma 1.1.

In the following section we shall construct an $\aleph_n$-free group which is $\aleph_{n+1}$-free for each $n \geq 1$ by making use of the stationary set $E = \{\lambda < \aleph_n : \mathrm{cf}(\lambda) = \aleph_{n-1}\}$. For the case $n = 1$, E is the set of all limit ordinals less than $\aleph_1$ and our construction reduces to the Griffith-Hill construction.

2. **The construction of κ-free groups.** A *smooth chain* of groups is an ascending chain $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\nu \subseteq \cdots$ $\nu < \mu$ such that for any limit ordinal $\lambda < \mu$, $A_\lambda = \bigcup_{\nu < \lambda} A_\nu$.

2.1 **Lemma** *Let μ be a limit ordinal and let A be the union of a smooth chain of free groups $\{A_\nu : \nu < \mu\}$ such that*

(*) *for each successor ordinal $\nu < \mu$, A_σ / A_ν is free whenever $\nu < \sigma < \mu$.*

Let E = the set of limit ordinals $\lambda < \mu$ such that $A_{\lambda+1}/A_\lambda$ is not free. Then

(i) *if E is not stationary in μ, then A is free and for any successor ordinal ν, A/A_ν is free;*

(ii) *if E is stationary in μ, $\mathrm{cf}(\mu)$ is uncountable, and for each $\nu < \mu$, the cardinality of $A_\nu < \mathrm{cf}(\mu)$, then A is not free.*

Proof. Let $\kappa = \mathrm{cf}(\mu)$. (i) Since E is not stationary in μ there is a closed unbounded subset $C = \{\rho_\sigma : \sigma < \kappa\}$ of μ which does not intersect E. We may assume that the ρ_σ's are strictly increasing. Thus there is a subsequence $\{A_{\rho_\sigma} : \sigma < \kappa\}$ of $\{A_\nu : \nu < \mu\}$ which forms a smooth chain whose union is A. (It is smooth because C is closed and has union A because C is unbounded.) Since $E \cap C = \emptyset$, the subsequence has the property that for all $\sigma < \tau < \kappa$, $A_{\rho_\tau}/A_{\rho_\sigma}$ is free. It is then evident that A is free and that A/A_{ρ_σ} is free for all $\sigma < \kappa$. The second part of the conclusion of (i) follows immediately from (*).

(ii) Suppose in contradiction to the conclusion of (ii) that A is free and write $A = \bigoplus_{i \in I} \mathbf{Z}_i$ where $\mathbf{Z}_i \cong \mathbf{Z}$ for all i. For any subset $J \subseteq I$ let $A^{(J)}$ denote $\bigoplus_{i \in J} \mathbf{Z}_i$; thus $A^{(I)} = A$. Now consider $C = \{\lambda < \mu : A_\lambda = A^{(J_\lambda)}$

for some $J_\lambda \subseteq I\}$. If we can prove that C is a closed, unbounded subset of μ, then we will be done for $C \cap E$ will be nonempty, but if $\lambda \in C$, $A_\lambda = A^{(J)}$ is certainly a direct summand of A, while on the other hand, $\lambda \in E$ implies $A_{\lambda+1}/A_\lambda$ is not free, so certainly A_λ is not a direct summand of A. Now C is unbounded since if $\bar{\lambda} = \sup S$ where $S \subseteq C$, then $A_{\bar{\lambda}} = A^{(J)}$ where $J = \bigcup\{J_\lambda : \lambda \in S, A_\lambda = A^{(J_\lambda)}\}$. To see that C is closed in μ consider any $\nu_0 < \mu$. There is a $J_0 \subseteq I$ such that $A_{\nu_0} \subseteq A^{(J_0)}$ and the cardinality of $J_0 = \aleph_0 +$ (cardinality of A_{ν_0}). Since the cardinality of $A_{\nu_0} < \kappa = \mathrm{cf}(\mu) \geq \aleph_1$ there is a ν_1 with $\nu_0 < \nu_1 < \mu$, such that $A^{(J_0)} \subseteq A_{\nu_1}$. Repeating the argument we obtain a strictly increasing sequence of ordinals $\nu_0 < \nu_1 < \nu_2 < \cdots$ and an increasing chain of subsets $J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots$ such that $A_{\nu_n} \subseteq A^{(J_n)} \subseteq A_{\nu_{n+1}}$ for all $n < \omega$. Letting $\lambda = \sup\{\nu_n : n < \omega\}$ and $J = \bigcup\{J_n : n < \omega\}$ we see that λ is a limit ordinal $> \nu_0$ such that $A_\lambda = A^{(J)}$, i.e. $\lambda \in C$. Thus, C is unbounded and the proof of 2.1 is complete.

Our main theorem is the following.

2.2 **Theorem.** *Let κ be a regular infinite cardinal such that there is a κ-free group which is not κ^+-free. Then there is a κ^+-free group which is not κ^{++}-free.*

We immediately obtain by induction the following corollary of the theorem.

2.3 **Corollary.** *For each $n < \omega$, there is an $\aleph_n$-free group of cardinality $\aleph_n$ which is not free.*

Proof of Theorem 2.2. We shall prove the theorem by constructing a smooth chain $\{A_\nu : \nu < \kappa^+\}$ of free groups of cardinality κ such that the conditions of Lemma 2.1(ii) are satisfied (with $\mu = \kappa^+$). Then $A = \bigcup_{\nu<\kappa^+} A_\nu$ will not be free by 2.1(ii) but it will be κ^+-free since every subgroup of A of cardinality $\leq \kappa$ is contained in some A_ν since κ^+ is regular. Let $E = \{\lambda < \kappa^+ : \mathrm{cf}(\lambda) = \kappa\}$, as in Lemma 1.1. We construct the groups A_ν by transfinite induction. Let A_0 be any free group of cardinality κ. Suppose that A_ν has been constructed for each $\nu < \mu < \kappa^+$ such that $\{A_\nu : \nu < \mu\}$ satisfies condition (*) of Lemma 2.1 and such that

(**) for any limit ordinal $\lambda < \mu$, $A_{\lambda+1}/A_\lambda$ is free if and only if $\lambda \notin E$.

The construction of A_μ is divided into three cases.

Case I. μ is a limit ordinal. In this case, let $A_\mu = \bigcup_{\nu<\mu} A_\nu$. By Lemma 1.1, $E \cap \mu$ is not stationary in μ; therefore by Lemma 2.1 (i), A_μ is free and (*) holds for $\{A_\nu : \nu < \mu + 1\}$.

Case II. μ is of the form $\delta + 1$ where $\delta \notin E$. In this case let

$A_\mu = A_\delta \oplus \mathbf{Z}$. It is obvious that A_μ is free and that (*) and (**) hold for $\{A_\nu: \nu < \mu + 1\}$.

Case III. μ is of the form $\lambda + 1$ where $\lambda \in E$. In this case we will use the hypothesis of the theorem to construct $A_{\lambda+1}$ so that $A_{\lambda+1}/A_\lambda$ is not free but $A_{\lambda+1}/A_\nu$ is free for every successor ordinal $\nu < \lambda$. By the definition of E, $\mathrm{cf}(\lambda) = \kappa$. As in the proof of 2.1(i) choose a subsequence $\{\rho_\sigma: \sigma < \kappa\}$ of λ such that $\{A_{\rho_\sigma}: \sigma < \kappa\}$ is a smooth chain whose union is A_λ and such that $A_{\rho_\tau}/A_{\rho_\sigma}$ is free whenever $\sigma < \tau < \kappa$. For convenience let us denote A_{ρ_σ} by $\tilde{A}_\sigma$. For the rest of the proof it will suffice to restrict our attention to the chain $\{\tilde{A}_\sigma: \sigma < \kappa\}$.

By hypothesis there is a free group F with a subgroup B such that F/B is κ-free of cardinality κ but is not free. We may assume F and B have rank κ; say $F = \bigoplus_{\nu<\kappa}\mathbf{Z}_\nu$, with $\mathbf{Z}_\nu \cong \mathbf{Z}$. For any ordinal $\alpha < \kappa$ let $F_\alpha = \bigoplus_{\nu<\alpha}\mathbf{Z}_\nu$ and let $B_\alpha = F_\alpha \cap B$. Note that $F/B_\alpha = (F_\alpha/B_\alpha) \oplus \bigoplus_{\nu\geq\alpha}\mathbf{Z}_\nu$ and $F_\alpha/B_\alpha \cong F_\alpha + B/B \subseteq F/B$, from which it follows that F_α/B_α and F/B_α are free since F/B is κ-free and $F_\alpha + B/B$ is generated by $<\kappa$ elements. Let $\tilde{B}_\alpha = \tilde{A}_0 \oplus B_\alpha$. We define a strictly increasing subsequence $\{\alpha_\sigma: \sigma < \kappa\}$ of κ and for each $\sigma < \kappa$ an isomorphism $f_\sigma: \tilde{B}_{\alpha_\sigma} \to \tilde{A}_\sigma$ such that $f_\sigma \subseteq f_\tau$ whenever $\sigma < \tau < \kappa$. Let $\alpha_0 = 0$ and let $f_0 =$ identity. If α_ν and f_ν have been defined for each $\nu < \sigma$, and σ is a limit ordinal let $\alpha_\sigma = \bigcup\{\alpha_\nu: \nu < \sigma\}$ and let $f_\sigma = \bigcup\{f_\nu: \nu < \sigma\}$. If $\sigma = \delta + 1$, let α_σ be the least ordinal $\alpha < \kappa$ such that $B_\alpha/B_{\alpha_\delta}$ has rank $=$ rank of $\tilde{A}_\sigma/\tilde{A}_\delta$. (This is possible since rank of $B_{\beta+1}/B_\beta \leq 1$ for any $\beta < \kappa$ while rank of $\tilde{A}_\sigma/\tilde{A}_\delta < \kappa =$ rank of B/B_β for any $\beta < \kappa$. Note that $\alpha_\sigma > \alpha_\delta$ since rank $\tilde{A}_\sigma/\tilde{A}_\delta \geq 1$.) Then since $B_{\alpha_\sigma}/B_{\alpha_\delta}$ and $\tilde{A}_\sigma/\tilde{A}_\delta$ are free and are of the same rank, it is clear that f_δ extends to an isomorphism. $f_\sigma: \tilde{B}_{\alpha_\sigma} \to \tilde{A}_\sigma$. Now $\alpha_\sigma \geq \sigma$ for every $\sigma < \kappa$ since α_σ is strictly increasing. Thus $\bigcup_{\sigma<\kappa} B_{\alpha_\sigma} = \bigcup_{\sigma<\kappa} B_\sigma = B \cap (\bigcup_{\sigma<\kappa} F_\sigma) = B \cap F = B$; and $\bigcup_{\sigma<\kappa} f_\sigma$ is an isomorphism: $\tilde{A}_0 \oplus B \to A_\lambda$. Now choose $A_{\lambda+1}$ to be a free group containing A_λ such that there is an isomorphism $f: \tilde{A}_0 \oplus F \to A_{\lambda+1}$ which is an extension of $\bigcup_{\sigma<\kappa} f_\sigma$. Then $A_{\lambda+1}/A_\lambda \cong F/B$ is not free but for every $\sigma < \kappa$, $A_{\lambda+1}/\tilde{A}_\sigma \cong F/B_{\alpha_\sigma}$ is free. Since for any successor ordinal $\nu < \lambda$ there is a $\sigma < \kappa$ such that $A_\nu \subseteq \tilde{A}_\sigma$, and by (*) $\tilde{A}_\sigma/A_\nu$ is free, we see that $A_{\lambda+1}/A_\nu$ is free and therefore (*) holds for $\{A_\nu: \nu < \mu + 1\}$. Since (**) holds by construction, the proof of the theorem is complete.

As another immediate corollary of the theorem we have the following.

2.4 **Corollary.** *If for some regular cardinal $\aleph_\alpha$ there is an $\aleph_\alpha$-free group which not $\aleph_{\alpha+1}$-free, then for any $n < \omega$ there is an $\aleph_{\alpha+n}$-free group which is not $\aleph_{\alpha+n+1}$-free.*

3. **Other consequences.** In this section we present some other consequences of our techniques. These involve some concepts from model theory and set theory for which we shall only give references. First we observe that Theorem 2.2 may be generalized in the following way.

3.1 **Theorem.** *Let κ be a regular infinite cardinal such that there is a κ-free group which is not κ^+-free. Then there is a nonfree group of cardinality κ^+ which is $L_{\infty\kappa^+}$ equivalent to a free group.*

Proof. We shall make use of an algebraic criterion for a group to be $L_{\infty\kappa^+}$ equivalent to a free group which was proved in [2], viz. A is $L_{\infty\kappa^+}$ equivalent to a free group if and only if A is κ^+-free and every subgroup of A of cardinality $\leq\kappa$ is contained in a κ^+-pure subgroup of cardinality $\leq\kappa$. (Recall that B is a κ^+-pure subgroup of A if whenever $B \subseteq C \subseteq A$ and C/B has cardinality $\leq\kappa$, then B is a direct summand of C.) Now the κ^+-free group A which we constructed in the proof of 2.2 is the union of a chain $\{A_\nu : \nu < \kappa^+\}$ such that each $A_{\nu+1}$ is a direct summand of A_μ whenever $\nu + 1 \leq \mu < \kappa^+$. Thus $A_{\nu+1}$ is κ^+-pure in A. Since every subgroup of A of cardinality $\leq\kappa$ is contained in some $A_{\nu+1}$, the criterion is satisfied and A is $L_{\infty\kappa^+}$-equivalent to a free group.

As an immediate corollary we obtain the following which proves a conjecture of Hill [11], also proved independently by Hill [12].

3.2 **Corollary.** *For every $n < \omega$, there is a nonfree group of cardinality $\aleph_n$ which is $L_{\infty\aleph_n}$-equivalent to a free group.*

The remainder of our observations result from the fact that our construction of a κ-free group which is not κ^+-free requires a stationary set $E \subseteq \kappa$ which is not stationary in any ordinal less than κ and such that for each $\gamma \in E$ we know that there is a $\mathrm{cf}(\gamma)$-free group which is not $\mathrm{cf}(\gamma)^+$-free. An inductive argument yields the following theorem, announced in [3].

3.3 **Theorem.** *Suppose that κ is a regular infinite cardinal such that for every regular infinite cardinal $\gamma \leq \kappa$, there is a stationary set $E_\gamma \subseteq \gamma$ such that $E_\gamma \cap \mu$ is not stationary in μ for any limit ordinal $\mu < \gamma$. Then there is a nonfree group of cardinality κ which is $L_{\infty\kappa}$-equivalent to a free group.*

It is an open problem whether one may prove in Zermelo-Fraenkel set theory that $\aleph_{\omega+1}$ satisfies the hypothesis of 3.3.

If we make an additional set-theoretic assumption, namely Gödel's axiom of constructibility (usually abbreviated "$V = L$"; see, for example, [13]) then we can prove the existence of stationary sets of the kind we are interested in. In fact, Jensen has proved the following theorem [14].

3.4. **Theorem (Jensen).** *Assume* $V = L$. *Let* κ *be a regular cardinal which is not weakly compact. Then there is a stationary* $E \subseteq \kappa$ *such that* $E \cap \mu$ *is not stationary in* μ *for any* $\mu < \kappa$ *and moreover such that for every* $\gamma \in E$, $\mathrm{cf}(\gamma) = \aleph_0$.

Then by the remark following 3.3 we see that we obtain the following result which was first proved by Gregory [6].

3.5 **Theorem (Gregory).** *Assume* $V = L$. *Let* κ *be a regular cardinal which is not weakly compact. Then there exists a* κ*-free group which is not* κ^+*-free.*

In contrast to 3.5 let us state without proof the following result of Mekler [15], observed independently by Gregory (for the definition of weakly and strongly compact cardinals see, for example [1, pp. 289–290], where what we call "strongly compact" is there called "compact").

3.6 **Theorem (Mekler).** *Suppose* κ *is weakly compact* (*resp. strongly compact*). *Then every* κ*-free group is* κ^+*-free* (*resp. is free*).

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