ABSTRACT. Let $X$ be a finite tree. It is shown that $X$ is order-isomorphic to the prime spectrum of a Bezout domain $R$ such that every localization of $R$ is a maximal valuation ring.

Let $R$ be a Bezout domain and let $\text{spec } R$ denote the set of prime ideals of $R$ considered as a partially ordered set under inclusion. It is well known that the localizations of $R$ at prime ideals are valuation rings and that $\text{spec } R$ is a tree: for each $s \in \text{spec } R$, $\{x \in \text{spec } R: x \leq s\}$ is totally ordered. The trees of the form $\text{spec } R$, $R$ a Bezout domain, have been completely characterized by Lewis [L], although the localizations $R_M$ in this construction are in general badly behaved. In this paper we show that every finite tree with a unique minimal element is order-isomorphic to $\text{spec } R$, for some Bezout domain $R$ which is locally a maximal valuation ring.

The construction in this paper is a generalization of an example of Barbara Osofsky which appears in [M2]. I would like to thank her for sending me the details of her construction with some suggestions for generalization. Also I am grateful to Tom Shores who carefully worked out some helpful facts about maximally complete fields.

Main theorem. Let $X$ be a finite tree. Then there exists a Bezout domain $R$ such that (1) $\text{spec } R$ is order-isomorphic to $X$, and (2) the localizations of $R$ at prime ideals are maximal valuation rings.

Proof. The procedure will be to exhibit $R$ as the intersection of a finite number of maximal valuation rings. Let $\mathcal{C} = \{C_0, C_1, \ldots, C_n\}$ denote the set of maximal chains indexed so that $\text{length } (C_i) \leq \text{length } (C_{i-1})$, $1 \leq i \leq n$. (As usual, the length of a chain is one less than its cardinality.) For each maximal chain $C_i$, we find a maximal valuation ring $\mathfrak{O}_i$, whose rank is equal to the length of the chain $C_i$ and whose quotient field is an alg-
braicly closed field \( K \) (fixed for all \( i \)). More precisely, we first construct \( \mathcal{O}_0 \), whose rank is equal to the length of \( C_0 \); then each \( \mathcal{O}_i \) will be an appropriate localization of \( \mathcal{O}_0 \). In order to produce maximal valuation rings \( V_i, 0 \leq i \leq n \), such that \( R = V_0 \cap V_1 \cap \cdots \cap V_n \) has the correct prime spectrum, we will define automorphisms \( \alpha_i \) of \( K \) in such a way that \( V_i = \alpha_i(\mathcal{O}_{i}) \) has the appropriate number of prime ideals in common with \( \mathcal{O}_0 \).

In what follows, we let \( C \) denote the field of complex numbers; \( \mathbb{Q} \) is the additive ordered group of the rationals; and \( \mathbb{Q}^+ \) is the set of positive rationals. Let \( k \) be the length \( (C_0, \), a longest chain in \( X \).

We begin by constructing \( \mathcal{O} = \mathcal{O}_0 \), which is to be a valuation ring of rank \( k \). Define a valuation \( v \) on \( \mathbb{C}(x_1, x_2, \cdots, x_k) \), the quotient field of the complex numbers, with \( k \) indeterminates adjoined, to be the valuation induced by \( v(x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}) = (a_1, a_2, \cdots, a_k) \in G \), where \( G \) is the direct sum of \( k \) copies of \( \mathbb{Q} \), ordered lexicographically. Let \( K \) be the maximal completion of \( \mathbb{C}(x_1, x_2, \cdots, x_k) \) with respect to \( v \); we denote the extension of \( v \) to \( K \) by \( v \) also. Now \( \mathcal{O} \), the valuation ring of \( K \), is a maximal valuation ring since \( K \) is maximally complete \([S, p. 45]\). Also \( K \) is algebraically closed. (The latter statement follows from \([S, Theorem 12, p. 57 \) and Theorem 11, p. 54].)

Since \( v \) behaves properly on monomials with integer exponents, it must do so on monomials with rational exponents, that is \( v(x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}) = (a_1, a_2, \cdots, a_k) \), for all \( a_i \in \mathbb{Q}^+ \). Now for each \( j \leq k \), \( P_j = \{rx^j \mid r \in \mathcal{O}, \ a \in \mathbb{Q}^+ \} \) is a prime ideal of \( \mathcal{O} \). (Note that an element \( r \in \mathcal{O} \) is in \( P_j \) if and only if \( v(r) \) has one of its first \( r \) coordinates positive.) Also, because the rank of \( G \) is \( k \), these together with \( P_0 = (0) \) are all the prime ideals of \( \mathcal{O} \). (See \([S, Corollary, p. 15] \) for more details here.) To see that \( P_{j-1} \subseteq P_j \), note that \( v(x_{j-1}/x_j) = (0, 0, 0, \cdots, 0, 1, -1, 0, 0, \cdots, 0) \), which is positive in the lexicographic ordering. Thus \( x_{j-1}/x_j \in \mathcal{O} \), so \( x_j^{a_j} \in P_j \) for all \( 1 \leq j \leq k \) and \( a \in \mathbb{Q}^+ \).

Thus we have found a valuation ring \( \mathcal{O} \) with quotient field \( K \) and valuation \( v: K \to G \) (the direct sum of \( k \) copies of \( \mathbb{Q} \)), satisfying these properties:

1. \( K \) contains \( \mathbb{C}(x_1, x_2, \cdots, x_k) \), where \( x_1, x_2, \cdots x_k \) are indeterminates over \( \mathbb{C} \).

2. The map \( v: K \to G \) reads exponents of monomials, that is, for any set of \( k \) rational numbers \( \{a_1, a_2, \cdots, a_k\} \), we have \( v(x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}) = (a_1, a_2, \cdots, a_k) \).

3. The prime ideals of \( \mathcal{O} \) are \( \{P_j \mid 0 \leq j \leq k\} \) where \( P_0 = (0) \) and
Let \( P_j = \{ rx^a | r \in \mathcal{O}, a \in \mathbb{Q}^+ \} \); also \( P_i \subseteq P_j \) if \( i \leq j \).

We now pause to establish notation which will be fixed for the rest of the paper. As above, suppose (i) \( X \) has maximal chains \( C_0, C_1, \cdots, C_n \), (ii) for each \( 0 \leq i \leq n \), the length of \( C_i \) is \( k_i \) (a positive integer), and (iii) \( k_0 \geq k_1 \geq \cdots \geq k_n \). Write (iv) \( C_i = \{ c_{i0}, c_{i1}, \cdots, c_{ik_i} \} \), where \( C_{io} < c_{i1} < \cdots < c_{ik_i} \).

We use an induction procedure to define automorphisms \( \alpha_i \) of \( K \). Let \( \alpha_0 \) be the identity automorphism. To define \( \alpha_i \), choose some maximal chain \( C_m \) with \( 0 \leq m \leq i - 1 \), such that among all the maximal chains \( C_0, C_1, \cdots, C_{i-1} \), the chain \( C_m \) has the largest number of elements in common with \( C_i \). (Clearly the minimal element of \( X \) is in every \( C_i \).) Let \( r \) be the (unique) integer such that \( c_{ir} = c_{mj} \) for all \( j < r \) but \( c_{ir} \neq c_{mr} \). (Then, since \( X \) is a tree, \( c_{ij} \neq c_{mj} \) for all \( r \leq j \leq k_m \).) Define \( \alpha_i \) by \( \alpha_i(x_j) = \alpha_m(x_j) \) if \( 0 < j < r \) and \( \alpha_i(x_j) = x_j + i \) if \( r \leq j \leq k \), and extend \( \alpha_i \) to an automorphism (still called \( \alpha_i \)) of \( K \) fixing \( C \). (Here \( i \) is regarded as a constant polynomial.) This is possible since the \( x_j \) are part of a transcendence basis for \( K \) over \( C \), and \( K \) is algebraically closed. Notice that the \( \alpha_i \) have been defined so that

\[
(4) \quad \text{For each } j, \text{ if } m \text{ is the least integer for which } c_{mj} = c_{ij}, \text{ then } \alpha_i(x_j) = \alpha_m(x_j) = x_j + m. \quad \text{In particular } \alpha_i(x_{ik_i}) = x_{ik_i} + i.
\]

\( (5) \) Claim: If \( \alpha_i(x_j) \in \alpha_m(P_i) \), for \( 0 \leq i, m \leq n \) and \( 1 \leq l, t \leq k \), then \( \alpha_i(x_l) = \alpha_m(x_l) \) and \( l \leq t \).

**Proof.** By (4) above, \( \alpha_i(x_j) = \alpha_m(x_j) + d \), for some integer \( d \). Thus \( \alpha_m(x_l + d) = \alpha_m(x_l) + d = \alpha_i(x_l) \in \alpha_m(P_t) \), and so \( x_l + d \in P_t \). If \( d \neq 0 \), then \( x_l + d \) is a unit in \( \mathcal{O} \); hence \( d \) must be 0 and \( l \leq t \).

Let \( \mathcal{O}_i \) be the localization \( \mathcal{O}_{P_i} \) for \( 0 \leq i \leq n \), and let \( V_i = \alpha_i(\mathcal{O}_i) \).

Our next project is to investigate the prime ideals of \( R = V_0 \cap V_1 \cap \cdots \cap V_n \).

Notice that the \( V_i \) are irredundant in the intersection for \( R \). For if not, suppose \( V_0 \cap V_1 \cap \cdots \cap V_{m-1} \cap V_{m+1} \cap \cdots \cap V_n \subseteq V_m \). Let \( t = k_m \) and set \( N_m = \alpha_m(P_t) \), the maximal ideal of \( V_m \). Now set \( z = x_t + m = \alpha_m(x_t) \); \( z \) is an element of \( N_m \), so \( 1/z \notin V_m \). Thus \( 1/z \notin V_i \) for some \( i \neq m \), whence \( z \in N_i = \alpha_i(P_k) \), the maximal ideal of \( V_i \). By (5), \( \alpha_m(x_t) = \alpha_i(x_t) \) and \( t \leq k_i \), and using (4) we deduce \( c_{mt} = c_{it} \). Since \( X \) is a tree, \( c_{mt} = c_{ij} \) for all \( 0 \leq j \leq i \). Now \( C_m \) has only \( t \) elements; thus \( C_m \subseteq C_i \), a contradiction.

By [K, Theorem 107, p. 78], \( R \) is Bezout and the maximal ideals of \( R \) are precisely \( {}|N_j \cap R, 0 \leq j \leq n| \), where \( N_j \) is the maximal ideal of \( V_j \); also \( V_j \) is the localization of \( R \) at \( N_j \cap R \). It easily follows that every prime ideal of \( R \) is of the form \( Q \cap R \), where \( Q \) is a prime ideal of some \( V_j \), that is:
The prime ideals of $R$ are of the form $\alpha_i(P_j) \cap R$ with $0 \leq i \leq n$, $0 \leq j \leq k_i$. (Note that these ideals need not be distinct for different $i$.)

Each localization of $R$ is a localization of some $V_i$ and the latter is isomorphic to the maximal valuation ring $\hat{\mathcal{O}}_i$. Therefore the localizations of $R$ at prime ideals are maximal valuation rings.

To complete the proof, we define a function $\phi$ from $X$ to $\text{spec } R$ by $\phi(c_{ij}) = \alpha_i(P_j) \cap R$; we will show $\phi$ is an order isomorphism. First we check well-definedness of $\phi$. It suffices to show that if $c_{lj} = c_{st}$ for some $s < l$, then $\phi(c_{lj}) = \phi(c_{st})$. Since $X$ is a tree, $j = t$ and $c_{lh} = c_{sh}$ for all $h \leq j$. As in (4), let $m$ be the smallest nonnegative integer so that $c_{mj} = c_{lj}$; then $\alpha_m(x_i) = \alpha_i(x_i)$ and so $\alpha_m(P_j) = \alpha_i(P_j)$. Also $m \leq s$ and $m$ is the smallest index with $c_{mj} = c_{sj}$, so $\alpha_m(x_i) = \alpha_s(x_i)$. Thus $\phi(c_{lj}) = \phi(c_{st})$.

The function $\phi$ is an isomorphism. For suppose $\phi(c_{ij}) = \phi(c_{kl})$, that is $\alpha_i(P_j) \cap R = \alpha_k(P_l) \cap R$, then $\alpha_i(x_i) = \alpha_k(x_l) \in \alpha_i(P_j) \cap R$. By (5), this implies $\alpha_i(x_i) = \alpha_k(x_l)$ and $t \leq j$. Similarly, from $\alpha_i(x_i) \in \alpha_k(P_l)$, we see $\alpha_i(x_i) = \alpha_k(x_l)$ and $t = j$. Choose $m$ smallest so that $c_{mj} = c_{lj}$ and choose $h$ smallest so that $c_{hj} = c_{lj}$. Then $\alpha_m(x_i) = \alpha_m(x_l) = x_i + m$, and $\alpha_h(x_l) = \alpha_h(x_l) = x_i + h$; thus $h = m$ and $c_{ij} = c_{lj} = c_{lt}$.

We claim $\phi$ is an order isomorphism. If $c_{lt} \leq c_{ij}$ for some $0 \leq l$, $i \leq n$, $0 \leq t \leq k_l$, $0 \leq j \leq k_i$, then since $X$ is a tree, $c_{lt} = c_{it}$ and $t \leq j$. But $\phi(c_{lt}) = \phi(c_{it}) = \alpha_i(P_j) \cap R$ and $\phi(c_{ij}) = \alpha_i(P_j) \cap R$. Now $t \leq j$ implies $P_{it} \subseteq P_j$ and $\alpha_i(P_j) \cap R \subseteq \alpha_i(P_j) \cap R$. Thus $\phi(c_{lt}) \leq \phi(c_{ij})$. This completes the proof.

Remarks. It is unknown whether the Bezout domains $R$ constructed here enjoy the property: every finitely generated $R$-module is a direct sum of cyclic modules. In [M1], Matlis shows the property is possessed by rings of the type we have constructed if each nonzero prime ideal is contained in a unique maximal ideal.

The procedure used in our construction can be used to produce certain infinite trees. For example, we can certainly construct a maximal valuation ring $\hat{\mathcal{O}}_0$ such that $\text{spec } \hat{\mathcal{O}}_0$ looks like $\mathbb{N}^{-1} = \{1/n \cup \{0\}$, where $n$ ranges over the natural numbers. By altering this $\hat{\mathcal{O}}_0$ as outlined above, we can display any tree (1) which has only finitely many maximal chains $C_i$, and (2) each $C_i$ looks like $\mathbb{N}^{-1}$.

In [L], Lewis shows $X \sim \text{spec } R$ for some Bezout domain $R$ if and only if (1) $X$ has a unique minimal element; (2) if $x, y, z \in X$ and $x \leq z$, $y \leq z$, then $x \leq y$ or $y \leq x$ (3) if $x, y \in X$ and $x < y$, then there exists $z, w \in X$ with $x \leq z < w \leq y$ and no elements of $X$ lie between $z$ and $w$; and (4)
every totally ordered subset of $X$ has a least upper bound and greatest lower bound. He shows that, for any partially ordered set $X$, there is a Bezout domain $R$ with $\text{spec } R$ order isomorphic to $X$ if and only if $X$ satisfies (1)–(4). In his construction, Lewis makes use of the Bezout domains constructed by Jaffard which appear in [J, Theorem 3, p. 78] and [O, p. 586]. Jaffard's Bezout domain is an overring of a group algebra $F[G]$ with the same quotient field as $F[G]$, where $F$ is a field and $G$ is a lattice ordered group. Thus, it can be shown that the domains Lewis produces are not locally maximal, if the partially ordered set has more than one element. (See for example [S, p. 36, 51].) We have still not answered the following question in the infinite case: is there a locally maximal Bezout domain for any partially ordered set $X$ satisfying (1)–(4)?

Added in proof. The author has shown that if a domain $R$ has the property that every finitely generated $R$-module is a direct sum of cyclics, then every nonzero prime ideal of $R$ is contained in a unique maximal ideal.

REFERENCES


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