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LOCALLY MAXIMAL BEZOUT DOMAINS

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ABSTRACT. Let X be a finite tree. It is shown that X is order-isomorphic to the prime spectrum of a Bezout domain R such that every localization of R is a maximal valuation ring.

Let R be a Bezout domain and let spec R denote the set of prime ideals of R considered as a partially ordered set under inclusion. It is well known that the localizations of R at prime ideals are valuation rings and that spec R is a tree: for each $s \in$ spec R, $\{x \in \text{spec } R: x \leq s\}$ is totally ordered. The trees of the form spec R, R a Bezout domain, have been completely characterized by Lewis [L], although the localizations R_M in this construction are in general badly behaved. In this paper we show that every *finite* tree with a unique minimal element is order-isomorphic to spec R, for some Bezout domain R which is locally a *maximal* valuation ring.

The construction in this paper is a generalization of an example of Barbara Osofsky which appears in [M2]. I would like to thank her for sending me the details of her construction with some suggestions for generalization. Also I am grateful to Tom Shores who carefully worked out some helpful facts about maximally complete fields.

Main theorem. *Let X be a finite tree. Then there exists a Bezout domain R such that* (1) spec R *is order-isomorphic to* X, *and* (2) *the localizations of R at prime ideals are maximal valuation rings.*

Proof. The procedure will be to exhibit R as the intersection of a finite number of maximal valuation rings. Let $\mathcal{C} = \{C_0, C_1, \cdots, C_n\}$ denote the set of maximal chains indexed so that length $(C_i) \leq$ length (C_{i-1}), $1 \leq i \leq n$. (As usual, the length of a chain is one less than its cardinality.) For each maximal chain C_i, we find a maximal valuation ring $\mathcal{O}_i$, whose rank is equal to the length of the chain C_i and whose quotient field is an alge-

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braically closed field K (fixed for all i). More precisely, we first construct $\mathcal{O}_0$, whose rank is equal to the length of C_0; then each $\mathcal{O}_i$ will be an appropriate localization of $\mathcal{O}_0$. In order to produce maximal valuation rings V_i, $0 \le i \le n$, such that $R = V_0 \cap V_1 \cap \cdots \cap V_n$ has the correct prime spectrum, we will define automorphisms α_i of K in such a way that $V_i = \alpha_i(\mathcal{O}_i)$ has the appropriate number of prime ideals in common with $\mathcal{O}_0$.

In what follows, we let $\mathbf{C}$ denote the field of complex numbers; $\mathbf{Q}$ is the additive ordered group of the rationals; and $\mathbf{Q}^+$ is the set of positive rationals. Let k be the length (C_0), a longest chain in X.

We begin by constructing $\mathcal{O} = \mathcal{O}_0$, which is to be a valuation ring of rank k. Define a valuation v on $\mathbf{C}(x_1, x_2, \cdots, x_k)$, the quotient field of the complex numbers, with k indeterminates adjoined, to be the valuation induced by $v(x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}) = (a_1, a_2, \cdots, a_k) \in G$, where G is the direct sum of k copies of $\mathbf{Q}$, ordered lexicographically. Let K be the maximal completion of $\mathbf{C}(x_1, x_2, \cdots, x_k)$ with respect to v; we denote the extension of v to K by v also. Now $\mathcal{O}$, the valuation ring of K, is a maximal valuation ring since K is maximally complete [S, p. 45]. Also K is algebraically closed. (The latter statement follows from [S, Theorem 12, p. 57 and Theorem 11, p. 54].)

Since v behaves properly on monomials with integer exponents, it must do so on monomials with rational exponents, that is $v(x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}) = (a_1, a_2, \cdots, a_k)$, for all $a_i \in \mathbf{Q}^+$. Now for each $j \le k$, $P_j = \{rx_j^a \mid r \in \mathcal{O},\ a \in \mathbf{Q}^+\}$ is a prime ideal of $\mathcal{O}$. (Note that an element $r \in \mathcal{O}$ is in P_j if and only if $v(r)$ has one of its first r coordinates positive.) Also, because the rank of G is k, these together with $P_0 = (0)$ are all the prime ideals of $\mathcal{O}$. (See [S, Corollary, p. 15] for more details here.) To see that $P_{j-1} \subseteq P_j$, note that $v(x_{j-1}/x_j) = (0, 0, 0, \cdots, 0, 1, -1, 0, 0, \cdots, 0)$, which is positive in the lexicographic ordering. Thus $x_{j-1}/x_j \in \mathcal{O}$, so $x_{j-1}^a \in P_j$ for all $1 \le j \le k$ and $a \in \mathbf{Q}^+$.

Thus we have found a valuation ring $\mathcal{O}$ with quotient field K and valuation $v: K \to G$ (the direct sum of k copies of $\mathbf{Q}$), satisfying these properties:

(1) K contains $\mathbf{C}(x_1, x_2, \cdots, x_k)$, where $x_1, x_2, \cdots x_k$ are indeterminates over $\mathbf{C}$.

(2) The map $v: K \to G$ reads exponents of monomials, that is, for any set of k rational numbers $\{a_1, a_2, \cdots, a_k\}$, we have $v(x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}) = (a_1, a_2, \cdots, a_k)$.

(3) The prime ideals of $\mathcal{O}$ are $\{P_j \mid 0 \le j \le k\}$ where $P_0 = (0)$ and

$P_j = \{rx_j^a \mid r \in \mathcal{O},\ a \in \mathbf{Q}^+\}$; also $P_i \subseteq P_j$ if $i \leq j$.

We now pause to establish notation which will be fixed for the rest of the paper. As above, suppose (i) X has maximal chains $C_0, C_1, \cdots, C_n$, (ii) for each $0 \leq i \leq n$, the length of C_i is k_i (a positive integer), and (iii) $k_0 \geq k_1 \geq \cdots \geq k_n$. Write (iv) $C_i = \{c_{i0}, c_{i1}, \cdots, c_{ik_i}\}$, where $C_{io} < c_{i1} < \cdots < c_{ik_i}$.

We use an induction procedure to define automorphisms α_i of K. Let α_0 be the identity automorphism. To define α_i, choose some maximal chain C_m with $0 \leq m \leq i-1$, such that among all the maximal chains $C_0, C_1, \cdots, C_{i-1}$, the chain C_m has the largest number of elements in common with C_i. (Clearly the minimal element of X is in *every* C_i.) Let r be the (unique) integer such that $c_{ij} = c_{mj}$ for all $j < r$ but $c_{ir} \neq c_{mr}$. (Then, since X is a tree, $c_{ij} \neq c_{mj}$ for all $r \leq j \leq k_m$.) Define α_i by $\alpha_i(x_j) = \alpha_m(x_j)$ if $0 < j < r$ and $\alpha_i(x_j) = x_j + i$ if $r \leq j \leq k$, and extend α_i to an automorphism (still called α_i) of K fixing $\mathbf{C}$. (Here i is regarded as a constant polynomial.) This is possible since the x_j are part of a transcendence basis for K over $\mathbf{C}$, and K is algebraically closed. Notice that the α_i have been defined so that

(4) For each j, if m is the least integer for which $c_{mj} = c_{ij}$, then $\alpha_i(x_j) = \alpha_m(x_j) = x_j + m$. In particular $\alpha_i(x_{k_i}) = x_{k_i} + i$.

(5) *Claim*: If $\alpha_i(x_l) \in \alpha_m(P_t)$, for $0 \leq i, m \leq n$ and $1 \leq l, t \leq k$, then $\alpha_i(x_l) = \alpha_m(x_l)$ and $l \leq t$.

Proof. By (4) above, $\alpha_i(x_l) = \alpha_m(x_l) + d$, for some integer d. Thus $\alpha_m(x_l + d) = \alpha_m(x_l) + d = \alpha_i(x_l) \in \alpha_m(P_t)$, and so $x_l + d \in P_t$. If $d \neq 0$, then $x_l + d$ is a unit in $\mathcal{O}$; hence d must be 0 and $l \leq t$.

Let $\mathcal{O}_i$ be the localization $\mathcal{O}_{P_i}$ for $0 \leq i \leq n$, and let $V_i = \alpha_i(\mathcal{O}_i)$. Our next project is to investigate the prime ideals of $R = V_0 \cap V_1 \cap \cdots \cap V_n$.

Notice that the V_i are irredundant in the intersection for R. For if not, suppose $V_0 \cap V_1 \cap \cdots \cap V_{m-1} \cap V_{m+1} \cap \cdots \cap V_n \subseteq V_m$. Let $t = k_m$ and set $N_m = \alpha_m(P_t)$, the maximal ideal of V_m. Now set $z = x_t + m = \alpha_m(x_t)$; z is an element of N_m, so $1/z \notin V_m$. Thus $1/z \notin V_i$ for some $i \neq m$, whence $z \in N_i = \alpha_i(P_{k_i})$, the maximal ideal of V_i. By (5), $\alpha_m(x_t) = \alpha_i(x_t)$ and $t \leq k_i$, and using (4) we deduce $c_{mt} = c_{it}$. Since X is a tree, $c_{mj} = c_{ij}$ for all $0 \leq j \leq t$. Now C_m has only t elements; thus $C_m \subseteq C_i$, a contradiction.

By [K, Theorem 107, p. 78], R is Bezout and the maximal ideals of R are precisely $\{N_j \cap R, 0 \leq j \leq n\}$, where N_j is the maximal ideal of V_j; also V_j is the localization of R at $N_j \cap R$. It easily follows that every prime ideal of R is of the form $Q \cap R$, where Q is a prime ideal of some V_j, that is:

(6) The prime ideals of R are of the form $\alpha_i(P_j) \cap R$ with $0 \leq i \leq n$, $0 \leq j \leq k_i$. (Note that these ideals need not be distinct for different i.)

Each localization of R is a localization of some V_i and the latter is isomorphic to the maximal valuation ring $\mathcal{O}_i$. Therefore the localizations of R at prime ideals are maximal valuation rings.

To complete the proof, we define a function ϕ from X to spec R by $\phi(c_{ij}) = \alpha_i(P_j) \cap R$; we will show ϕ is an order isomorphism. First we check well-definedness of ϕ. It suffices to show that if $c_{lj} = c_{st}$ for some $s < l$, then $\phi(c_{lj}) = \phi(c_{st})$. Since X is a tree, $j = t$ and $c_{lh} = c_{sh}$ for all $h \leq j$. As in (4), let m be the smallest nonnegative integer so that $c_{mj} = c_{lj}$; then $\alpha_m(x_j) = \alpha_l(x_j)$ and so $\alpha_m(P_j) = \alpha_l(P_j)$. Also $m \leq s$ and m is the smallest index with $c_{mj} = c_{sj}$, so $\alpha_m(P_j) = \alpha_s(P_j)$. Thus $\phi(c_{lj}) = \phi(c_{st})$.

The function ϕ is an isomorphism. For suppose $\phi(c_{ij}) = \phi(c_{lt})$, that is $\alpha_i(P_j) \cap R = \alpha_l(P_t) \cap R$, then $\alpha_l(x_t) \in \alpha_i(P_j) \cap R$. By (5), this implies $\alpha_i(x_t) = \alpha_l(x_t)$ and $t \leq j$. Similarly, from $\alpha_i(x_j) \in \alpha_l(P_t)$, we see $\alpha_i(x_j) = \alpha_l(x_j)$ and $t = j$. Choose m smallest so that $c_{mj} = c_{ij}$ and choose h smallest so that $c_{hj} = c_{lj}$. Then $\alpha_i(x_j) = \alpha_m(x_j) = x_j + m$, and $\alpha_l(x_j) = \alpha_h(x_j) = x_j + h$; thus $h = m$ and $c_{ij} = c_{lj} = c_{lt}$.

We claim ϕ is an order isomorphism. If $c_{lt} \leq c_{ij}$ for some $0 \leq l$, $i \leq n$, $0 \leq t \leq k_l$, $0 \leq j \leq k_i$, then since X is a tree, $c_{lt} = c_{it}$ and $t \leq j$. But $\phi(c_{lt}) = \phi(c_{it}) = \alpha_i(P_t) \cap R$ and $\phi(c_{ij}) = \alpha_i(P_j) \cap R$. Now $t \leq j$ implies $P_t \subseteq P_j$ and $\alpha_i(P_t) \cap R \subseteq \alpha_i(P_j) \cap R$. Thus $\phi(c_{lt}) \subseteq \phi(c_{ij})$. This completes the proof.

Remarks. It is unknown whether the Bezout domains R constructed here enjoy the property: every finitely generated R-module is a direct sum of cyclic modules. In [M1], Matlis shows the property is possessed by rings of the type we have constructed if each nonzero prime ideal is contained in a unique maximal ideal.

The procedure used in our construction can be used to produce certain infinite trees. For example, we can certainly construct a maximal valuation ring $\mathcal{O}_0$ such that spec $\mathcal{O}_0$ looks like $\mathbf{N}^{-1} = \{1/n\} \cup \{0\}$, where n ranges over the natural numbers. By altering this $\mathcal{O}_0$ as outlined above, we can display any tree (1) which has only finitely many maximal chains C_i, and (2) each C_i looks like $\mathbf{N}^{-1}$.

In [L], Lewis shows $X \sim$ spec R for some Bezout domain R if and only if (1) X has a unique minimal element; (2) if $x, y, z \in X$ and $x \leq z$, $y \leq z$, then $x \leq y$ or $y \leq x$; (3) if $x, y \in X$ and $x < y$, then there exists $z, w \in X$ with $x \leq z < w \leq y$ and no elements of X lie between z and w; and (4) every totally ordered subset of X has a least upper bound and greatest lower bound. He shows that, for any partially ordered set X, there is a Bezout domain R with spec R order isomorphic to X if and only if X satisfies (1)–(4). In his construction, Lewis makes use of the Bezout domains constructed by Jaffard which appear in [J, Theorem 3, p. 78] and [O, p. 586]. Jaffard's Bezout domain is an overring of a group algebra $F[G]$ with the same quotient field as $F[G]$, where F is a field and G is a lattice ordered group. Thus, it can be shown that the domains Lewis produces are not locally maximal, if the partially ordered set has more than one element. (See for example [S, p. 36, 51].) We have still not answered the following question in the infinite case: is there a locally maximal Bezout domain for any partially ordered set X satisfying (1)–(4)?

Added in proof. The author has shown that if a domain R has the property that every finitely generated R-module is a direct sum of cyclics, then every nonzero prime ideal of R is contained in a unique maximal ideal.

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