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ON A COEFFICIENT INEQUALITY FOR STARLIKE FUNCTIONS

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ABSTRACT. M. S. Robertson considered a coefficient inequality I for starlike functions that, if true, would imply a generalized Bieberbach coefficient inequality B for close to convex functions. An example is given of a starlike function whose coefficients do not satisfy coefficient inequality I.

Let S denote the collection of regular and univalent functions $f(z) = \sum_{n=1}^{\infty} A_n z^n$ in the unit disk U. Let S^* denote the subclass of S of functions f such that $f(U)$ is starlike with respect to the origin. Let K denote the close to convex functions in S, i.e. those functions $f \in S$ such that there exists a $g \in S^*$ and a real α such that $e^{i\alpha} zf'(z)/g(z) = p(z)$, where $p(z)$ is a regular function of positive real part in U.

In [2] and [3] M. S. Robertson considered a generalized Bieberbach inequality:

$$\big| n|A_n| - m|A_m| \big| \leq |n^2 - m^2|\, |A_1|, \tag{1}$$

for m and n nonnegative integers. In [2] Robertson showed that (1) holds for all functions convex in one direction (for definition, see [2]), and that (1) holds for all functions in K if $n - m$ is an even integer. In [1] J. A. Jenkins showed that the sharp upper bound for $3|A_3| - 2|A_2|$ is $(5.02\cdots)|A_1|$ when $f \in S$. It is still an open question whether (1) is true for all close to convex functions if $n - m$ is not an even integer. Robertson stated in [3] that (1) would hold for all functions in K if every starlike function $f(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $|z| < 1$, has coefficients which satisfy

$$|2b_n - b_2 b_{n-1}| \leq 2, \qquad n = 1, 2, 3, \cdots \qquad (b_0 = 0). \tag{2}$$

In this note we give an example of a function in S^* where (2) does not hold. Indeed, consider the function

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$$F_\alpha(z) = \left(\frac{1-z}{1+z}\right)^\alpha \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} b_n(\alpha) z^n, \qquad 0 \le \alpha \le 2,$$

where $b_2(\alpha) = 2(1-\alpha)$, $b_3(\alpha) = 3 - 4\alpha + 2\alpha^2$, $b_4(\alpha) = 4(3 - 5\alpha + 3\alpha^2 - \alpha^3)/3$, and $b_5(\alpha) = (15 - 28\alpha + 22\alpha^2 - 8\alpha^3 + 2\alpha^4)/3$. Since $F_0(z)$ and $F_2(z)$ map U univalently onto the complement of a radial ray, and $F_\alpha(z)$, $0 < \alpha < 2$, maps U univalently onto the complement of two radial rays, $F_\alpha \in S^*$ for each α, $0 \le \alpha \le 2$. If we let $T_n(\alpha) = 2b_n(\alpha) - b_2(\alpha)b_{n-1}(\alpha)$, then $T_4(0) = T_5(0) = T_5(1) = T_5(2) = 2$. It follows from the continuity of $\partial T_4(\alpha)/\partial\alpha$ that, since $\partial T_4(\alpha)/\partial\alpha\big|_{\alpha=0} > 0$, (2) does not hold for $F_\alpha(z)$ for some $\alpha > 0$. In fact for $\alpha \ne 1$ and $0 < \alpha < 2$, we have

$$2 < T_5(\alpha) = 2 + 4\alpha(2-\alpha)(1-\alpha)^2/3 \le T_5(1 - \sqrt{2}/2) = 7/3.$$

It is of interest to note that if one considers the function $F \in K$ defined by $zF'(z) = F_\alpha(z)P(z)$, where $P(z) = (1+z)/(1-z)$, as might be suggested by the method of proof used by Robertson in [2], that (1) holds for this function F.

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