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A PROOF OF BERNSTEIN'S THEOREM ON REGULARLY MONOTONIC FUNCTIONS

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ABSTRACT. A function is called "regularly monotonic" if it is of class C^∞ and each derivative is of a fixed sign (which may depend on the order of the derivative). We present a short proof of Bernstein's theorem on the analyticity of such functions.

This paper presents a short proof of Bernstein's theorem [1] on "regularly monotonic" functions. For background on this subject (and a presentation of Bernstein's original proof) we refer the reader to the brief survey paper by Boas [2], and the references cited there. The book [3] contains a proof of a special case of Bernstein's theorem. That proof for that special case partially motivated the proof we present here. Before giving our proof of Bernstein's theorem, we recall that a regularly monotonic function is a function of class C^∞ on a real interval (a, b) for which each derivative is of fixed sign on (a, b).

Theorem. *If $F(x)$ is regularly monotonic on (a, b), then $F(x)$ is analytic on (a, b).*

Proof. We assume for convenience that $a = -b < 0$. We will prove that the even part of $F(x)$, $f(x)$, is analytic at $x = 0$. An analogous proof holds for the odd part of $F(x)$, and the analyticity of $F(x)$ thereby follows. We begin by noting that $f(x)$ is itself regularly monotonic on $0 \le x < b$. The following lemma is strategic for our proof.

Lemma. *If $f^{(n)}(x) \le 0$, $f^{(n+1)}(x) \ge 0$, and $f^{(n+2)}(x) \ge 0$, then for $x \in [0, b)$, $|R_n(x)| \ge |R_{n+1}(x)|$, where $R_m(x)$ is the mth remainder in the Taylor expansion of f about $x = 0$.*

Proof of Lemma. Write the remainder as

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(1) $$R_n(x) = \frac{1}{(n-1)!}\int_0^x f^{(n)}(t)(x-t)^{n-1}dt.$$

A sufficient condition for the Lemma to be true is that

(2) $$-f^{(n)}(t) - \frac{(x-t)}{n} f^{(n+1)}(t) \geq 0$$

for t on $(0, x)$. Since $f^{(n+1)}(t) \geq 0$, replacing n by 1 in (2) yields a sufficient condition for (2) to be true; namely

$$-f^{(n)}(t) - (x-t)f^{(n+1)}(t) \geq 0,$$

or, if we write g for $f^{(n)}$,

(3) $$-g(t) - (x-t)g'(t) \geq 0.$$

By Rolle's theorem there exists a ξ on (t, x) such that

(4) $$-g(t) - (x-t)g'(\xi) = -g(x) \geq 0.$$

Since $g'' = f^{(n+2)} \geq 0$, (3) follows from (4) (Q.E.D. Lemma).

If instead of the sign sequence of the Lemma one has either of the sign triples +, +, + or +, +, −, the Lemma's conclusion is immediate. Only for polynomials (analytic functions) can the triple +, −, + occur when f is even. For, in this case, $f^{(n)}$ is even or odd. If $f^{(n)}$ is even, say, then $f^{(n+1)}(0) = 0$. By supposition, $f^{(n+2)}(x) \geq 0$ for $x \geq 0$. Thus $f^{(n+1)}(x) \geq 0$ for $x \geq 0$. But also by supposition, $f^{(n+1)}(x) \leq 0$ for $x \geq 0$. Thus $f^{(n+1)}(x) \equiv 0 \Rightarrow f$ is a polynomial. An analogous proof holds for $f^{(n)}$ odd. Since one of these sign triples is always attainable (perhaps after $f \to -f$), it follows that for f even

(5) $$|R_n(x)| \geq |R_{n+1}(x)| \quad \text{for } n = 0, 1, 2, \cdots,$$

for x on $[0, b)$.

Frequently f^n and f^{n+1} are both ≥ 0 (or both ≤ 0) (else eventually sign $(f^{(n)}) = (-1)^{n+K} \Rightarrow f$ is polynomial). Rewriting (1) as

(6) $$R_n(x) = \frac{x^n}{(n-1)!}\int_0^1 f^{(n)}(xt)(1-t)^{(n-1)}dt,$$

then for, say, $f^{(n_i)}$ and $f^{(n_i+1)} \geq 0$ and $x \geq 0$, one has

(7) $$0 \leq R_{n_i}(x) \leq \frac{x^{n_i}}{(n_i-1)!}\int_0^1 f^{(n_i)}(b't)(1-t)^{n_i-1}dt,$$

where $x < b' < b$, since $f^{(n_i)}$ is nondecreasing. Thus

$$0 \leq R_{n_i}(x) \leq (x/b')^{n_i} R_{n_i}(b'). \tag{8}$$

But $R_n(b')$ is bounded, whence $R_n(x) \to 0$ as $n \to \infty$. Since for $f(x)$ even, $R_n(x) = R_n(-x)$, this proves the analyticity of $f(x)$, the even part of $F(x)$. The analyticity of $F(x)$ itself then follows as initially indicated.

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