SHORTER NOTES

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ON SEQUENCES SPANNING A COMPLEX $l_1$ SPACE

LEONARD E. DOR

ABSTRACT. If $(f_n)$ is a bounded sequence in a complex Banach space $B$, and no subsequence of $(f_n)$ is weakly Cauchy, then a subsequence of $(f_n)$ is equivalent to the unit vector basis of the complex $l_1$ space.

In a recent paper [1], H. P. Rosenthal proves that if a bounded sequence $(f_n)$ of elements of a real Banach space has no weak Cauchy subsequence, then a subsequence of $(f_n)$ is equivalent to the unit vector basis of real $l_1$. The purpose of the present note is to prove the analogous result for complex Banach spaces. The proof follows the lines of [1]. Any $M \subseteq N$ is assumed to be infinite, and $(f_n)_{n \in M}$ has the natural meaning (as a sequence).

Let $S$ be the unit ball of $B^*$, $(f_n)$ as in the abstract, i.e. a sequence of uniformly bounded affine complex functions on $S$ with no pointwise convergent subsequence. Let $\mathcal{D}$ be the set of all pairs $(D_1, D_2)$ of open "rational" discs in the complex plane $\mathbb{C}$ such that $\text{diam}(D_1) = \text{diam}(D_2) < \frac{1}{2} \text{dist}(D_1, D_2)$, and let $\{(D_1^k, D_2^k); k \in \mathbb{N}\}$ be an enumeration of $\mathcal{D}$.

Claim. There are $k_0 \in \mathbb{N}$ and a subset $M \subseteq N$ so that for every $L \subseteq M$ there is $s \in S$ s.t. the sequence $(f_n(s))_{n \in L}$ has accumulation points both in $D_1^{k_0}$ and in $D_2^{k_0}$.

Otherwise, we can construct $N \supseteq M_1 \supseteq M_2 \supseteq \cdots$ so that for any $k \in \mathbb{N}$ and any $s \in S$ the sequence $(f_n(s))_{n \in M_k}$ has all its accumulation points outside $D_1^k$ or all of them outside $D_2^k$. Choose $m_1 < m_2 < \cdots$ s.t. $m_k \in M_k$.

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(k \in \mathbb{N})$, and let $L = \{m_k; k \in \mathbb{N}\}$. $(f_n)_{n \in L}$ does not converge pointwise on $S$ so there are $s \in S$ and $p \neq q$ in $C$ so that $p$ and $q$ are accumulation points of $(f_n(s))_{n \in L}$. Clearly, there is $k \in \mathbb{N}$ s.t. $p \in D^k_1$, $q \in D^k_2$, but then $(f_n(s))_{n \in M_k}$ has accumulation points in both $D^k_1$ and $D^k_2$.

Now let $k_0$ and $M$ be as above, let $\alpha$ be the center of $D^k_1$, $\beta$ that of $D^k_2$, and let $2\delta = \text{dist}(D^k_1, D^k_2)$. Multiplying all the functions $f_n$ by $|\beta - \alpha|/(\beta - \alpha)$, we may assume that $\beta - \alpha > 0$. For $n \in M$, let $A_n = \{s \in S; f_n(s) \in D^k_1\}$, and $B_n = \{s \in S; f_n(s) \in D^k_2\}$. Using Theorem 2 of [1], we get $L \subseteq M$ so that for any pair of disjoint finite sets $E, F \subseteq L$, $\bigcap_{n \in E} A_n \cap \bigcap_{n \in F} B_n \neq \emptyset$.

The sequence $(f_n)_{n \in L}$ is equivalent over the complex scalars to the unit vector basis of $l_1$: It is enough to show that $\|\sum_{k \in E} c_k f_k\| \geq (\delta/4) \sum_{k \in E} |c_k|$, for any finite $E \subseteq L$ and any choice of $c_k = a_k + ib_k (k \in E)$. Without loss of generality, $\sum_{k \in E} |a_k| \geq \sum_{k \in E} |b_k|$. Choose $s \in \bigcap_{a_k \geq 0} A_k \cap \bigcap_{a_k < 0} B_k$. Thus,

$$\left\|\sum_{k \in E} c_k f_k\right\| \geq \text{Re} \sum_{k \in E} c_k f_k \left(\frac{s-t}{2}\right)$$

$$\geq \sum_{k \in E} a_k \text{Re} f_k \left(\frac{s-t}{2}\right) - \sum_{k \in E} b_k \text{Im} f_k \left(\frac{s-t}{2}\right)$$

$$\geq \delta \sum_{k \in E} |a_k| - \frac{\delta}{2} \sum_{k \in E} |b_k| \geq \frac{\delta}{2} \sum_{k \in E} |a_k| \geq \frac{\delta}{4} \sum_{k \in E} |c_k|.$$}

The third inequality follows from the geometrically clear fact that $u \in D^k_1$, $v \in D^k_2$ imply:

$$\text{Re}(v-u) \geq 2\delta \quad \text{and} \quad \text{Im}(v-u) \leq \text{diam}(D^k_1) < \delta.$$}

**REFERENCE**


DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43220