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TRANSFORMATION GROUPS RESEMBLING THE ADJOINT REPRESENTATION

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ABSTRACT. If G is a compact, connected Lie group, the isotropy subgroups of the adjoint representation of G are connected and the dimension of the fixed point set of a maximal torus of G is equal to the the rank of G. Results similar to these are given when G acts differentiably on an integral cohomology sphere and has the adjoint representation as weak linear model. This is done by analyzing an induced action of the Weyl group of G.

1. Let G be a compact, connected Lie group. G acts on itself by conjugation, fixing the identity element and thereby giving rise to an orthogonal representation of G on the tangent space at the identity. This representation, called the *adjoint representation*, is completely described by its restriction to a maximal torus T of G, and this restriction is in turn completely described by the roots of G which are considered as vectors in the Lie algebra of T. The Lie algebra of T is contained in the Lie algebra of G; since it is fixed by T it is acted on by $NT/T = W$, where NT is the normalizer of T in G. The finite group W, called the *Weyl group* of G, permutes the roots of G so that if α is a root, so is $\sigma\alpha$ for any σ in W. W is generated by the orthogonal reflections through the hyperplanes perpendicular to the (nonzero) roots. If S_α is the reflection through the plane perpendicular to α and σ is an element of W, $\sigma S_\alpha \sigma^{-1}$ is reflection through the root $\sigma\alpha$. Every reflection in W is of the form S_β for some nonzero root β [2, p. 74]. $W(G)$ will also be used to denote the Weyl group of G.

Let $F(T)$ denote the fixed point set of T; G_x the subgroup of elements of G which fix x (called the *isotropy subgroup*); G_x^0 the connected component of G_x which contains the identity. If g is an element of G which fixes x but does not lie in G_x^0, then $g^{-1}Tg = k^{-1}Tk$ for some k in G_x^0 since all maximal tori of G_x^0 are conjugate, so gk^{-1} is an element of $W(G)_x$ when x is fixed by T. If α is a nonzero root and H_α is the plane fixed by S_α, the centralizer in G of the corank one subtorus of T corresponding to H_α

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is denoted by G_α. Since G_α is connected [3, pp. 26–31] and $W(G_\alpha)$ is generated by S_α, any point in $F(T)$ fixed by S_α is also fixed by G_α. Since $W(G)_x$ is generated by reflections which fix x, we have

Proposition 1. (a) *If $x \in F(T)$ is fixed by S_α, $G_\alpha \subset G_x$.*
(b) *The isotropy subgroups of the adjoint representation are connected.*
(c) *For $x \in F(T)$, $W(G)_x = W(G_x)$.*

2. We want to generalize these observations to differentiable actions on cohomology spheres. If M is a differentiable manifold, a *diffeomorphism reflection* on M is a diffeomorphism r such that (a) r reverses orientation and $r^2 = 1$; (b) the complement of the fixed point set of r is disconnected. We shall refer to diffeomorphism reflections simply as *reflections*. The fixed point set of a reflection on an integral cohomology sphere HS^n is a Z_2 cohomology sphere by Smith theory. However, a standard argument [6, p. 35] shows that in fact it is an integral cohomology sphere HS^{n-1}.

If W is a group which is the Weyl group of a semisimple Lie group G, the *rank* of W is defined as the rank of G (i.e. the dimension of a maximal torus of G). For such a group W we consider differentiable actions (W, ϕ, HS^k). Such an action is called *effective* if no element of W fixes every element of HS^k. Such an action is called *normal* if (a) each linear reflection in W is a reflection on HS^k; (b) no product of two distinct linear reflections in W acts trivially. Considering ϕ as a homomorphism of W into the group of diffeomorphisms of HS^k, we say that a reflection on HS^k is *proper* if it is the image under ϕ of a linear reflection in W.

Theorem 2. *If (W, ϕ, HS^k) $(k \geq 1)$ is a normal action, then it is effective and each reflection is proper.*

Proof. The proof is by induction on the rank of W. If rank $W = 1$, the theorem follows trivially from the definition. Let rank $W = n > 1$ and suppose that the theorem is true for rank W less than n. If S is a proper reflection there is a point x in HS^k which is fixed by S but which is not fixed by any other proper reflection since the proper reflections have distinct fixed point sets (which meet transversally). If t is any other element of W fixing such an x, then $t^{-1}St = S$. Hence any element of W acting trivially must lie in the center of W. If w is such an element, the eigenvalues of w are ± 1; but if w takes $+1$ as an eigenvalue, then w lies in a Weyl subgroup of W of less rank by Proposition 1(c). So we may assume that $w = -I$.

Lemma. *The element $-I$ in W can be written as the product of n mutually commuting distinct linear reflections where $n =$ rank W.*

Proof of Lemma. It is sufficient to prove this when W is the Weyl group of a simple Lie group. If $W = W(E_6)$, $W(A_q)$, $q \neq 1$, or $W(D_p)$, p odd, then $-I$ is not an element of W [2, p. 284]. For the remaining classical groups it is obvious, and since $W(B_4)$ is contained in $W(F_4)$ and $W(D_8)$ is contained in $W(E_8)$, we are left with G_2 and E_7. We use the notation of [2, pp. 250–275]. For G_2 reflect through roots in the set $\{\epsilon_1, \epsilon_2\}$. For E_7 use roots in the set $\{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2, \epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_4, \epsilon_5 - \epsilon_6, \epsilon_5 + \epsilon_6, \epsilon_7 - \epsilon_8\}$.

Returning to the proof of the theorem, we have $w = R_1 R_2 \cdots R_n$ acting trivially, $R_i R_j = R_j R_i$. Since each R_i reverses orientation, n must be even. But each R_i acts on the fixed point set of R_n, $F(R_n)$, and must reverse its orientation (for $i \neq n$) since it does so locally near a fixed point. Hence the case n even is also impossible.

If s in W is an improper reflection we have essentially the same situation. Since $s^2 = I$, either s lies in a Weyl subgroup of smaller rank or $s = -I$, so we may write $s = R_1 R_2 \cdots R_n$ in accordance with the Lemma above. Each R_i acts on $F(s)$ and reverses the orientation (or interchanges the two fixed points when $k = 1$). Since the product acts trivially on $F(s)$ we are done.

3. Let (G, ϕ, HS^n) be a differentiable action of a compact connected Lie group G on an integral cohomology sphere HS^n. Let $S(\phi)$ be the cw-system of ϕ [8]. $S(\phi)$ is defined as a collection of linear functionals on the Lie algebra of T which vanish on the Lie algebras of the connected corank one subtori of T which have fixed point sets of greater dimension than that of T. $S(\phi)$ includes the zero functional with multiplicity $dF(T) + 1$, where $dF(T)$ denotes the dimension of the fixed point set of T; $S'(\phi)$ denotes the collection of nonzero elements in $S(\phi)$. We assume for the remainder of this section that $S'(\phi) = S'(\mathrm{Ad}_G)$; the elements of this set then vanish on the hyperplanes perpendicular to the roots of G discussed in §1. The fixed point set of T is an integral cohomology sphere acted upon by $W(G)$. The fact that this is a normal action is proved in [8, 1.11] and will not be repeated here. It is an easy consequence of a theorem of W.-Y. Hsiang which states that Proposition 1(a) holds for differentiable actions (cf. [5, Proposition 3, p. 349]).

Since $W(G)_x$ is generated by the reflections which fix x [4, Theorem 1] and since these are now known to be proper reflections, it follows, just as in the proof of Proposition 1, that the isotropy subgroups are connected and that $W(G)_x = W(G_x)$ for x in $F(T)$. That T is a principal isotropy subgroup of ϕ now follows from [7, Theorem 3.3]. Since the $W(G)$ action is now

known to be effective, it follows from Theorem 3 of [4] that $dF(T) \geq \operatorname{rank} G - 1$. Thus we have

Theorem 3. *Let (G, ϕ, HS^n) be a differentiable action of a compact connected Lie group with $S'(\phi) = S'(\mathrm{Ad}_G)$. Then:*

(a) $dF(T) \geq \operatorname{rank}(G) - 1$;

(b) *All isotropy subgroups are connected*;

(c) *For x in $F(T)$, $W(G)_x = W(G_x)$.*

(d) $S(\phi) = S(\mathrm{Ad}_G + r$ *trivial copies*), *where* $r = dF(T) - \operatorname{rank} G + 1 = dF(G) + 1$.

Remark. Part of the proof of Theorem 3 of [4] has not yet appeared, namely the proof of Theorem 2′. In lieu of this, one can produce a weaker inequality in (a) above as follows: $W(G)$ contains subgroups isomorphic to $Z_2 \times \cdots \times Z_2 = (Z_2)^k$, where each copy of Z_2 is generated by a reflection in $W(G)$. If $-I$ is an element of $W(G)$, the Lemma shows that $k = \operatorname{rank} G$; otherwise k is at least $\frac{1}{2} \operatorname{rank} G$. One can apply the formula of Borel [1, Chapter XIII, Theorem 2.3] with Z_2 coefficients to these subgroups to show that $dF(T)$ is at least $\frac{1}{2} \operatorname{rank} G$.

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