ON INVERSE LIMITS OF HOMOTOPY SETS

PETER J. KAHN

ABSTRACT. An elementary proof is given that, under certain conditions on a space $F$, the homotopy set $[X, F]$ maps bijectively onto the inverse limit of homotopy sets determined by the finite subcomplexes of $X$. The only other satisfactory proof known requires the Brown representability theorem.

Throughout this note we deal only with based maps and based CW complexes. $X$ and $F$ will be such CW complexes, and $\{X_\alpha\}$ will be the set of finite subcomplexes of $X$, directed by inclusion. We assume that $F$ is connected and that each homotopy group of $F$ is finite.

Theorem 1. The natural map

$$[X, F] \xrightarrow{\pi^0_X} \lim_{\alpha} [X_\alpha, F]$$

is bijective.

This result is trivial when $X$ has dimension 0 or is a finite complex. Moreover, when $X$ is an increasing union of a sequence $\{X_n\}$ of subcomplexes such that $\pi^0_{X_n}$ is surjective for each $n$, then an inductive application of the homotopy extension property yields that $\pi^0_X$ is surjective. In particular, this gives surjectivity when $X$ is a countable CW complex. The problem is that no such straightforward argument seems to work for uncountable $X$.

Define $L^0_FX$ to be the inverse limit set in Theorem 1. The theorem may then be interpreted as saying that $L^0_F$ is a representable functor. This suggests a connection between Theorem 1 and the Brown representability theorem, and, indeed, Brown’s theorem has been used to prove Theo-
The purpose of this note is to present a direct, elementary proof.

Theorem 1 is generally false without some kind of finiteness condition on \( \pi_iF \) (e.g., see [4]), but the surjectivity of \( \pi^0_X \), for all \( X \), can be proved using an algebraic condition on \( F \) [1], [2]. The precise conditions for surjectivity are not yet well understood.

Now define
\[
L^1_FX = \lim_{\alpha} [X_\alpha \times I \cup X \times \partial I, F],
\]
where \( I \) is the unit interval. \((L^n_F \) can be defined analogously for \( n = 2, 3, \ldots, \) but they will not be needed.) We then have the following "Mayer-Vietoris" sequence of based sets
\[
[X \times I, F] \xrightarrow{\pi^1_X} L^1_FX \xrightarrow{i_0} [X, F] \xrightarrow{\pi^0_X} L^0_FX,
\]
in which \( \pi^1_X \) is induced by restriction, as in \( \pi^0_X \), and \( i_0 \) and \( i_1 \) are induced by the two natural inclusions of \( X \) into \( X_\alpha \times I \cup X \times \partial I \). Since \( i_0 \circ \pi^1_X = i_1 \circ \pi^1_X \) is a bijection, \( \pi^1_X \) is injective.

**Lemma 1.** The above sequence is exact at \( [X, F] \).

By this we mean that \( \pi^0_X \circ i_1 = \pi^0_X \circ i_0 \), which we take as obvious, and that if \( \pi^0_X(a) = \pi^0_X(b) \), then \( a = i_0(c), b = i_1(c) \) for some \( c \in L^1_FX \).

**Theorem 2.** \( \pi^0_X \) and \( \pi^1_X \) are surjective.

Therefore, \( \pi^1_X \) is bijective. Moreover, since \( \pi^1_X \) is surjective, it follows that \( i_0 = i_1 \), which, by Lemma 1, forces \( \pi^0_X \) to be injective. Thus, Theorem 2 implies Theorem 1.

By the remarks made earlier, we can obtain Theorem 2 for \( X \) provided that we can prove it for every skeleton of \( X \). It clearly holds for the 0-skeleton of \( X \). Thus, we need only

**Lemma 2.** Let \( Y \) be an \( n \)-dimensional CW complex, \( n \geq 1 \), and let \( X \) be a subcomplex of \( Y \) containing the \((n-1)\)-skeleton of \( Y \). If \( \pi^k_X \) is surjective, \( k = 0, 1 \), then so is \( \pi^k_Y \).

The proofs of Lemmas 1 and 2 are based on the following two elementary facts:

**Fact A.** An inverse limit of nonempty finite sets is nonempty.

**Fact B.** Let \( Y \) be a CW complex, \( X \) a subcomplex of \( Y \), and \( F \) as before. Let \( g : X \to F \) be any map. If \( Y \setminus X \) consists of finitely many
cells, then there are only finitely many homotopy classes rel $X$ of extensions $Y \to F$ of $g$.

Let $[Y, F]$ denote this set of homotopy classes. Its finiteness is a result of elementary obstruction theory. Fact B applies when $X$ is empty, and it still holds, of course, if "rel $X$" is deleted. Fact A is a direct consequence of König's lemma on finitely branching trees.

Proof of Lemma 1. For each $\alpha$, there are maps $j_0, j_1: [X_\alpha \times I \cup X \times \partial I, F] \to [X, F]$ determined, as before, by the two natural inclusions of $X$. Since $\pi_X^0(a) = \pi_X^0(b)$, there exists, for each $\alpha$, a $c_\alpha$ satisfying $j_0(c_\alpha) = a$, $j_1(c_\alpha) = b$. Letting $J_\alpha$ be the set of all such $c_\alpha$, we note that the $J_\alpha$'s, together with restriction maps, form an inverse system. Since $X_\alpha \times I$ is a finite CW complex, Fact B implies that $J_\alpha$ is finite. Fact A then produces a $\phi \in \lim J_\alpha \to L^1_F X$ with the required properties. □

Proof of Lemma 2. The proof for $\pi_X^1$ is the same as that for $\pi_X^0$, and so for notational simplicity we do only the latter.

Case 1. $Y \setminus X$ has only finitely many cells.

We redefine $L^0_F X$ and $L^0_F Y$ by passing to cofinal subsets of $\{X_\alpha\}$ and $\{Y_\alpha\}$: Namely, we use only $X_\alpha$ containing the boundaries of all cells in $Y \setminus X$, and we use only $Y_\alpha$ of the form $X_\alpha \cup (Y \setminus X)$. This enables us to define the restriction $L^0_F Y \to L^0_F X$.

Fix $\alpha$, and note that the commutative diagram

$$
\begin{array}{ccc}
[Y_\alpha, F] & \to & [X_\alpha, F] \\
\downarrow & & \downarrow \\
[Y, F] & \to & [X, F]
\end{array}
$$

has, by homotopy extension, the following exactness property: If $\phi_\alpha \in [Y_\alpha, F]$ and $\psi \in [X, F]$ restrict to $\psi_\alpha \in [X_\alpha, F]$, then there exists a $\phi \in [Y, F]$ restricting to both $\phi_\alpha$ and $\psi$. Let $J_\alpha$ be the set of all such $\phi$. By Fact B, $J_\alpha$ is finite.

If $\{\phi_\alpha\} \in L^0_F Y$ is arbitrary, $\{\psi_\alpha\} \in L^0_F X$ obtained from it by restriction, and if $\psi \in [X, F]$ satisfies $\pi_X^0(\psi) = \{\psi_\alpha\}$, which $\psi$ exists by hypothesis, then the collection of all $J_\alpha$ that we obtain as above for these $\phi_\alpha$, $\psi_\alpha$ and $\psi$ form a system directed by inclusion. By Fact A, $\bigcap J_\alpha$ is nonempty. Each member $\phi$ satisfies $\pi_X^0(\phi) = \{\phi_\alpha\}$.

Case 2. The general case. So far we have not used the full strength of Fact B: Namely, that it applies to homotopy classes rel $X$. We now use this.
We begin as before by choosing $\{\phi_\alpha\} \in L^0_F Y$, restricting to $\{\psi_\alpha\} \in L^0_F X$, and pulling back to a class $\psi \in [X, F]$ represented by some $g: X \to F$. Let $\mathcal{J}$ index the cells of $Y \setminus X$, and let $\{\sigma\}$ be the collection of its finite subsets, directed by inclusion. Define $Y(\sigma) = \bigcup i \in \sigma \{i\} \subseteq Y \setminus X$, and let $p_\sigma: [Y(\sigma), F] \to [Y(\sigma), F]$ be the standard projection. In Case 1, we showed that $\text{image } p_\sigma$ contains a nonempty finite set $J_{\sigma}$ such that $\pi^0_\sigma(J_{\sigma}) = \{\phi_\alpha\} | Y(\sigma)$. Let $K_{\sigma} = p_\sigma^{-1}(J_{\sigma}) \subseteq [Y(\sigma), F]_g$. Using Fact A and arguing as before, we conclude that there is a class $\{\phi'_\sigma\} \in \varprojlim K_{\sigma}$. For each singleton $\sigma$, let $J'_{\sigma}$ represent $\phi'_\sigma$, let $\phi' \in [Y, F]_g$ be given by $\bigcup J'_{\sigma}$, and let $\phi \in [Y, F]$ be given by the same map. For any $\sigma$, $\phi' | [Y(\sigma), F]_g = \phi'_\sigma$. Letting $Y_\alpha$ be any finite subcomplex of $Y$, we choose $\sigma$ so that $Y_{\sigma} \supset Y_\alpha$, and we obtain the desired conclusion that

$$\pi^0_\sigma(\phi) = \{\phi | Y_\alpha\} = \{p_\sigma(\phi'_\sigma) | Y_\alpha\} = \{\phi_\alpha\}. \quad \square$$

REFERENCES


2. A. Deleanu, Remark on the Brown-Adams representability theorem, Syracuse University, 1971 (mimeo).


DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14850

Current address: Mathematisches Institut der Universität Heidelberg, 6900 Heidelberg 1 im Neuenheimer Feld 288, Heidelberg, West Germany