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ON ITERATES OF CONVOLUTIONS

SHAUL R. FOGUEL[1]

ABSTRACT. *Let μ be a signed measure, of total variation one, on a locally compact Abelian group.* We study in this note the ideal $I = \{\tau: \tau \ll m$ *and* $\|\mu^n * \tau\| \to 0\}$ *where* m *is the Haar measure.*

Let G be a locally compact Abelian group, Γ its dual group and m its Haar measure. Throughout this note we shall use the notation of [5].

Let μ be a fixed signed measure (not necessarily continuous with respect to m), let ν be the total variation of μ. We shall assume throughout:

Assumption. $\nu(G) = 1$.

Definition 1. $I = \{\tau: \tau \ll m$ and $\|\mu^n * \tau\| \to 0$ as $n \to \infty\}$.

Definition 2. *Let J be the closed ideal, of signed measures, generated by $(\mu_A - \nu(A)\mu) * \eta$ where $\eta \ll m$, A is a measurable subset of G and μ_A is the restriction of μ to A.*

Lemma 1. $J \subset I$.

Proof. Let $A \subset G$ satisfy $0 < \nu(A) < 1$. Put $\mu_1 = \nu(A)^{-1}\mu_A$, $\mu_2 = (1 - \nu(A))^{-1}(\mu - \mu_A)$. Then $\mu = \nu(A)\mu_1 + (1 - \nu(A))\mu_2$ and, by Lemma 2.1 of [3], $\|\mu^n * (\mu_1 - \mu_2)\| \to 0$. Now

$$\mu_1 - \mu_2 = \nu(A)^{-1}(1 - \nu(A))^{-1}(\mu_A - \nu(A)\mu),$$

hence $\|\mu^n * (\mu_A - \nu(A)\mu)\| \to 0$. Clearly this remains true if $\nu(A) = 0$ or $\nu(A) = 1$ too. Thus, $(\mu_A - \nu(A)\mu) * \eta \in I$ whenever $\eta \ll m$. $\square$

Theorem 1. *Let $\gamma \in \Gamma$; the following conditions are equivalent:*

(a) *γ vanishes on J.*

(b) $(d\mu/d\nu)(x) = (x, \gamma)\hat{\mu}(\gamma)$ *a.e.* ν.

(c) $|\hat{\mu}(\gamma)| = 1$.

(d) *γ vanishes on I.*

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Proof. (a) $\Rightarrow$ (b). If $(\hat{\mu}_A(\gamma) - \nu(A)\hat{\mu}(\gamma)) \cdot \hat{\eta}(\gamma) = 0$ for every $\eta \ll m$ and every set A, then $\int_A (-x, \gamma)\, \mu(dx) = \int_A \hat{\mu}(\gamma)\, \nu(dx)$. Now $\int_A (-x, \gamma)\, \mu(dx) = \int_A (-x, \gamma)(d\mu/d\nu)(x)\nu(dx)$ from which (b) follows.

(b) $\Rightarrow$ (c). Note that $|(x, \gamma)| = 1$ and $|(d\mu/d\nu)(x)| = 1$ a.e. ν.

(c) $\Rightarrow$ (d). If $\|\mu^n * \tau\| \to 0$ where $|\hat{\mu}(\gamma)| = 1$, then $|\hat{\tau}(\gamma)| = |\widehat{\mu^n * \tau}(\gamma)| = 0$.

(d) $\Rightarrow$ (a). Follows from Lemma 1. □

Definition 3. $E = E(\mu) = \{\gamma\colon \gamma \in \Gamma$ *and* $\hat{\tau}(\gamma) = 0$ *for every* $\tau \in I\} = \{\gamma\colon \gamma \in \Gamma$ *and* $\hat{\tau}(\gamma) = 0$ *for every* $\tau \in J\}$.

Let us use now the notion of a set of spectral synthesis, see [5, 7.1.4, p. 158]. Note that by (c) of Theorem 1, E is a closed set. Now if $\gamma_0 \in E$ and $\gamma_0 + \gamma \in E$ it follows from (b) of Theorem 1 that

$$(x, \gamma_0)\hat{\mu}(\gamma_0) = (x, \gamma + \gamma_0)\hat{\mu}(\gamma + \gamma_0) \quad \text{a.e. } \nu$$

hence $(x, \gamma) =$ const. a.e. ν. Conversely, if $\gamma_0 \in E$ and $(x, \gamma) =$ const. a.e. ν then

$$\hat{\mu}(\gamma_0 + \gamma) = \int (-x, \gamma_0)(-x, \gamma)\mu(dx) = \text{const. } \hat{\mu}(\gamma_0)$$

and $\gamma_0 + \gamma \in E$ too.

Thus E is a coset of Γ and by [5, 7.5.2, part (d), p. 170] E is a set of spectral synthesis. Thus

Theorem 2. $I = J = \{\tau\colon \tau \ll m$ *and* $\hat{\tau}(\gamma) = 0$ *for all* $\gamma \in E\}$.

Remark. From part (b) of Theorem 1 follows that if $d\mu/d\nu$ is not a multiple of (x, γ), for some γ, a.e. ν then $J = I = L_1(G)$.

Theorem 3. *If* ν *is not supported on a set of the form* $\{x\colon (x, \gamma_0) = \text{const.}\}$ *for some* $\gamma_0 \in \Gamma$, *then* E *is either empty or contains one point.*

Proof. Let $\gamma_1, \gamma_2 \in E$; then $(x, \gamma_1) = \hat{\mu}(\gamma_1)$ and $(x, \gamma_2) = \hat{\mu}(\gamma_2)$ a.e. ν. Hence $(x, \gamma_1 - \gamma_2) =$ const. a.e. ν which contradicts our assumption. □

Remark. Assume the contradiction of Theorem 3 and $d\mu(x)/d\nu = c(x, \gamma)$ a.e. ν. Then $\hat{\mu}(\gamma) = c$ and $\gamma \in E$.

Corollary 1. *If* $\mu \geq 0$ *and the support of* μ *is not contained in any set of the form* $\{x\colon (x, \gamma_0) = \text{const.}\}$, *then* $\|\mu^n * \tau\| \to 0$, *as* $n \to \infty$, *whenever* $\tau \ll m$ *and* $\tau(G) = 0$.

Proof. If $\mu \geq 0$ then $0 \in E$ (0 is the character which is identically equal to 1). Thus $E = \{0\}$ by Theorem 3. □

Remark. For $G = R^1$ this corollary was proved in [4, Theorem 7, part (b), p. 11], by a probabilistic argument.

The following corollary was suggested to us by John Baxter.

Corollary 2. *If the support of* μ *is not contained in any set of the form* $\{x: (x, y) = \text{const.}\}$ *and* f *is a measurable bounded function such that* $\mu * f = f$ *a.e.* m, *then* $f = \text{const.}$ *a.e.* m.

Proof. By Corollary 1 if $\tau \ll m$ and $\tau(G) = 0$, then

$$|\tau * f| = |(\tau * \mu^n) * f| \leq \|f\|_\infty \|\tau * \mu^n\|_1 \to 0.$$

Thus $f = \text{const. a.e. } m$. □

Remarks. Corollary 2 is proved in [2, Chapter XI, §2, p. 351]. It is proved also in [1] for non-Abelian groups but under the restrictive assumption that μ induces a conservative random walk. Also in [1] it is shown that this corollary implies that the only bounded measure τ, with $\tau * \mu = \tau$ is a multiple of Haar measure if G is compact and is zero otherwise. Finally note that Corollary 2 is a generalization of the classical result that a bounded harmonic function is a constant: take $G = R^n$ and μ a uniform distribution on a sphere around the origin.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455

DEPARTMENT OF MATHEMATICS, HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, ISRAEL