THE MEASURE OF THE INTERSECTION OF ROTATES OF A SET ON THE CIRCLE

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ABSTRACT. Let $S$ be a set of real numbers modulo 1 of Lebesgue measure less than 1. It is shown that for every $\epsilon > 0$ and for large $k$, there exist translates $S + y_1, \ldots, S + y_k$ of $S$ such that the measure of their intersection is less than $\epsilon k$.

1. Let $U$ be the group of real numbers modulo 1, and $S$ a subset of Lebesgue measure $\mu(S) < 1$. Given real numbers $y_1, \ldots, y_k$, write $\mu(y_1, \ldots, y_k)$ for the measure of the intersection of the $k$ translates $S + y_1, S + y_2, \ldots, S + y_k$. Finally, denote by $\phi(k)$ the infimum of $\mu(y_1, \ldots, y_k)$ over all $k$-tuples $y_1, \ldots, y_k$. Erdős, Rubel and Spencer had conjectured that

$$\lim_{k \to \infty} \phi(k)^{1/k} = 0. \tag{1}$$

In the present note we shall prove this conjecture.

The convergence expressed by (1) is not uniform with respect to the sets $S$. In fact, it can be shown that for $0 < \alpha < 1$, $\epsilon > 0$ and $k \geq 1$, there exist sets $S$ with $\mu(S) = \alpha$ and $\phi(k)^{1/k} > \alpha - \epsilon$.

2. Since $\mu(S) < 1$, the set $S$ is contained in a countable union of intervals whose total measure is less than 1. In fact, this is true even with intervals of the type $a < x < b$ with rational endpoints $a, b$. Hence we may assume that $S$ itself is a countable union of such intervals.

Using the easily established relation

$$\int_U \mu(y_1, \ldots, y_m, z_1 + x, \ldots, z_n + x) dx = \mu(y_1, \ldots, y_m) \mu(z_1, \ldots, z_n),$$

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3 We are considering intervals modulo 1. Hence if $\{x\}$ denotes the fractional part of $x$, the interval $a < x < b$ consists of numbers $x$ modulo 1 with $\{x - a\} < \{x - b\}$. 

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one sees that $\phi(m + n) \leq \phi(m)\phi(n)$. Hence if $t$ is any positive integer, we have $\phi(jt) \leq \phi(t)^j$ ($j = 1, 2, \cdots$), and if $k$ is a large integer with $jt < k \leq (j + 1)t$, then $\phi(k) \leq \phi(jt)\phi(k - jt) \leq \phi(jt) \leq \phi(t)^j$ and

$$\phi(k)^{1/k} \leq \phi(t)^{j/k} \leq \phi(t)^{1/t} - (1/k).$$

Therefore the limit superior of $\phi(k)^{1/k}$ as $k \to \infty$ cannot exceed $\phi(t)^{1/t}$.

Thus in order to prove (1), it will suffice to show that for every $\epsilon > 0$ there is an integer $t$ with

$$(2) \phi(t)^{1/t} < \epsilon.$$

3. Write $\mu(S) = \mu$, and choose $\delta > 0$ so small that

$$(3) 2\delta < 1 - \mu \quad \text{and} \quad (\delta/(1 - \mu - \delta))(1 - \mu - \delta) < \epsilon.$$

We may write $S = S_1 \cup S_2$, where $S_1$ is a finite union of intervals $a < x < b$ with rational endpoints, and where $\mu(S_2) < \delta$.

Let $r$ be a common denominator of the endpoints of the intervals contributing to $S_1$. Choose an integer $s$ with $s > 1/\delta$, and put

$$(4) t = rs, \quad \nu = 1/t.$$

Let $\chi(x)$ be the characteristic function of $S$, and write

$$I_\nu(y) = \int_y^{y+\nu} \chi(x)\,dx.$$

Lemma. The function

$$J_\nu(z) = I_\nu(z + \nu)I_\nu(z + 2\nu) \cdots I_\nu(z + t\nu)$$

satisfies $J_\nu(z) \leq (\nu^t)$.\]

To prove the Lemma, we observe that $S_1$ consists of a finite number (in fact less than $r$) intervals $E$ of the type $(u/r) \leq x < (u + 1)/r$ with integral $u$. For each such interval $E$ contained in $S_1$, let $E'$ be the enlarged interval $(u/r) - (1/t) \leq x < (u + 1)/r$. Let $S_1'$ be the union of the intervals $E'$ so obtained. It is clear that

$$(5) \text{if } x + w \in S_1 \text{ with } 0 \leq w \leq \nu, \text{ then } x \in S_1'.$$

For each interval $E$ above we have $\mu(E') = \mu(E) + (1/t)$, and hence we have

$$\mu(S_1') < \mu(S_1) + (r/t) = \mu(S_1) + (1/s) < \mu(S) + \delta = \mu + \delta.$$ \[Now $S_1'$ is a disjoint union of intervals $(v/t) \leq x < (v + 1)/t$ with integral $v$. If, say, it is a disjoint union of $p$ such intervals, then $\mu(S_1') = p/t$ and hence

$$(6) p = t\mu(S_1') < t(\mu + \delta).$$
Exactly \( q = t - p \) of the numbers \( z + \nu, z + 2\nu, \ldots, z + t\nu \) lie outside \( S'_1 \); let these be the numbers \( z + m_1\nu, z + m_2\nu, \ldots, z + m_q\nu \). Since each integral \( I_\nu(y) \) is always \( \leq \nu \), we have

\[
I_\nu(z) \leq \nu^p I_\nu(z + m_1\nu) \cdots I_\nu(z + m_q\nu).
\]

(7)

Now \( I_\nu(z + m_\nu) \) is the integral of \( \chi(x) \) over the interval \( z + m_i\nu \leq x < z + (m_i + 1)\nu \) \((i = 1, \ldots, d)\). These intervals are disjoint from each other. Furthermore, since \( z + m_i\nu \notin S'_1 \), (5) implies that these intervals are disjoint from \( S_1 \). Therefore if \( S^-_1 \) is the complement of \( S_1 \), we have

\[
I_\nu(z + m_1\nu) + \cdots + I_\nu(z + m_q\nu) \leq \int_{S^-_1} \chi(x) \, dx \leq \mu(S^-_2) < \delta.
\]

By the arithmetic-geometric inequality, the product of the \( q \) integrals on the left is \( < (\delta/q)^q \), and (7) yields

\[
I_\nu(z) < \nu^p (\delta/q)^q = \nu^t(\delta t/q)^q.
\]

From (6) we have \( q = t - p > \ell(1 - \mu - \delta) \), whence

\[
(\delta t/q)^q < (\delta/(1 - \mu - \delta))^q < (\delta/(1 - \mu - \delta))^{\ell(1 - \mu - \delta)} < \epsilon^t
\]

by (3), and the Lemma is proved.

4. The desired inequality (2) follows at once from the Lemma by observing that

\[
\phi(t) \leq \nu^{-t} \int_{\nu}^{2\nu} dy_1 \cdots \int_{t\nu}^{(t+1)\nu} dy_t \mu(-y_1, \ldots, -y_t)
= \nu^{-t} \int_U dx \int_{\nu}^{2\nu} dy_1 \cdots \int_{t\nu}^{(t+1)\nu} dy_t \chi(x + y_1) \cdots \chi(x + y_t)
= \nu^{-t} \int_U dx \int U_{\nu}(x + \nu) I_\nu(x + 2\nu) \cdots I_\nu(x + t\nu)
= \nu^{-t} \int_U I_\nu(x) \, dx < \nu^{-t}(\epsilon\nu)^t = \epsilon^t.
\]