A NOTE ON LOCALLY FINITE GROUP ALGEBRAS

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ABSTRACT. We obtain an injectivity condition for group algebras which is equivalent to local finiteness.

1. Introduction. Several authors have studied the effect of various injectivity conditions on group algebras. Connell [1] showed that if the group algebra $F[G]$ is self-injective, then $G$ is locally finite; Renault [4] improved this result by showing that $G$ is, in fact, finite. The following question arises: what weakening of self-injectivity coincides with local finiteness? This note provides one answer.

We will say that a ring $R$ with 1 is principally (right) self-injective if any right $R$-module map from a principal right ideal of $R$ into $R$ can be lifted to all of $R$. Notice that this definition is the usual Baer criterion for self-injectivity if we omit the two occurrences of the word "principal".

If $M$ is a right $R$-module and $S$ is a subset of $R$, then $l_M(S) = \{ m \in M | ms = 0 \ \forall s \in S \}$ is the left annihilator of $S$ in $M$. Left actions give rise to right annihilators. If $R = M$ we say that a left ideal $L$ of $R$ is a left annihilator if $L = l_R(S)$ for some subset $S \subseteq R$; equivalently, $L = l_R(r_R(L))$. We will drop subscripts when the context is clear.

If $G$ is a group and $F$ is a field, then $F[G]$ will denote the set of all infinite formal sums $\sum f_g g$ with $f_g \in F$ and $g \in G$. Under pointwise addition $F[G]$ becomes a right $F[G]$-module containing $F[G]$.

Finally we can state our result.

Theorem. The following properties are equivalent:
2. $G$ is locally finite.
4. Every principal left ideal of $F[G]$ is an annihilator.

The equivalence of 2 and 4 is of particular interest. It might be regarded as a first approximation to the following longstanding conjecture: if every element of $F[G]$ is a zero-divisor or invertible then $G$ is locally finite.

2. A proof. Crucial to all proofs of local finiteness is

Lemma [3, p. 105]. Let $g_1, \ldots, g_n$ be a finite number of elements of $G$, and let $H = \langle g_1, \ldots, g_n \rangle$ be the subgroup of $G$ they generate. Then

$$\{ r \in F[G] | (g_i - 1)r = 0 \text{ for } i = 1, \ldots, n \} = \begin{cases} 0 & \text{if } H \text{ is infinite,} \\ \left( \sum_{h \in H} b \right) F[G] & \text{if } H \text{ is finite.} \end{cases}$$

We proceed to the Theorem:

$1 \Rightarrow 2$. It suffices to prove that $< H, x >$ is finite whenever $H$ is a finite subgroup of $G$ and $x \in G$. (One can then argue local finiteness by inducting on the number of generators of a finitely generated subgroup of $G$.) Set $s = \sum_{b \in H} h$. The Lemma shows either $< H, x >$ is finite or $(x - 1)s a = 0 \Rightarrow sa = 0 \forall a \in F[G]$. In the latter case the $F[G]$-map $\phi: (x - 1)s F[G] \rightarrow F[G]$ given by $((x - 1)sa)\phi = sa$ is well defined. By hypothesis $3 d \in F[G] \Rightarrow (x - 1)s \phi = d(x - 1)s$, i.e. $(1 - d(x - 1))s = 0$. Since both $d(x - 1)$ and any annihilator of $s$ are in the augmentation ideal of $F[G]$, 1 is in the augmentation ideal, a contradiction. Thus $< H, x >$ is finite.

$2 \Rightarrow 3$. Let $t_i, v_i$ be a left transversal for the finite subgroup $< \text{supp } a > = H$ in $G$. If $\sum_{t_i \in H} a \in F[G]$ with $b_i \in F[H]$ then $S = \{ i \in | b_i \cdot a \neq 0 \}$ is finite. Since $H$ is finite, $\sum_{t_i \in H} a \in F[G] \Rightarrow < \sum_{t_i \in H} b_i \cdot a > = (\sum_{t_i \in H} b_i) \cdot a$.

$3 \Rightarrow 4$. It is enough to show that $l_{F[H]}(r_{F[G]}(a)) = F[G]a$. The inclusion "$\supseteq\" is trivial. If $b$ is in the double annihilator then the $F$-linear map $\tau: aF[G] \rightarrow F$ given by $\tau(ar) = tr(br)$ is well defined. (Here, $tr$ of an element in $F[G]$ denotes the coefficient of 1.) Lift $\tau$ to an $F$-linear map on $F[G]$. Writing $a = \sum h a_h b$, a finite sum, we have

$$b = \sum_{g \in G} tr(b g^{-1}) g = \sum_{g} tr(a g^{-1}) g = \sum_{g} \left( \sum_{h} a_h b g^{-1} \right) g$$

$$= \sum_{h} \left( \sum_{g} a_h tr(b g^{-1}) \right) g = \sum_{h} \left( \sum_{g} tr(b g^{-1}) g^{-1} \right) a_h b = \left( \sum_{g \in G} tr(y^{-1}) y \right) a \in F[G]a.$$

$4 \Rightarrow 1$. It is easy to see that any ring $R$ is principally right self-injective iff for each $a \in R$, $Ra = l(r(a))$ [2, Theorem 1].
REFERENCES


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