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A MINIMAL DECAY RATE FOR SOLUTIONS OF STABLE nTH ORDER HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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ABSTRACT. In this paper we establish the existence of an envelope function (depending only on n and $\alpha > 0$) which provides a pointwise bound on the size of any normalized solution y of any homogeneous nth order differential equation with constant coefficients for which the roots of the corresponding characteristic polynomial have real parts which do not exceed $-\alpha$. An explicit representation for this envelope is obtained in the special case where these roots are further constrained to be real valued.

1. **Introduction.** For $n = 1, 2, \cdots$ and for $\alpha > 0$ we define $L_{n\alpha}$ to be the set of all possible complex valued solutions of all possible nth order homogeneous differential equations

$$[(D - \lambda_1)(D - \lambda_2) \cdots (D - \lambda_n)]y(t) = 0, \qquad t \geq 0,\ D = d/dt, \tag{1}$$

for which

$$\operatorname{Re} \lambda_i \leq -\alpha, \qquad i = 1, 2, \cdots, n. \tag{2}$$

When $y \in L_{n\alpha}$, $y(t)$ decays to zero as $t \to +\infty$ so that

$$\|y\| = \max\{|y(t)|: t \geq 0\} \tag{3}$$

provides a well-defined measure of the overall size of y.

In this paper we show that the parameters n, α, and $\|y\|$ are sufficient to characterize a minimal decay rate for such a function y.

2. **The existence of the envelope function.** This minimal decay rate can be bounded by means of an envelope function in the following sense.

Theorem 1. *For each $n = 1, 2, \cdots$ there exists a nonincreasing function g_n: $[0, +\infty) \to (0, 1]$ with*

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(4) $$g_n(t) \downarrow 0 \quad as\ t \longrightarrow +\infty$$

such that the inequality

(5) $$|y(t)| \le \|y\| \cdot g_n(\alpha t), \qquad t \ge 0,$$

holds for every $y \in L_{n\alpha}$.

Proof. We define $g_n\colon [0, +\infty) \to (0, 1]$ by

(6) $$g_n(t) = \sup\{|y(t)|\colon y \in L_{n1}, \|y\| = 1\}, \qquad t \ge 0,$$

and show that g_n satisfies the requirements of the theorem. Using (6) and the definition of $L_{n\alpha}$ we see that

(7) $$\begin{aligned} g_n(\alpha t) &= \sup\{|y(\alpha t)|\colon y \in L_{n1}, \|y\| = 1\} \\ &= \sup\{|y(t)|\colon y \in L_{n\alpha}, \|y\| = 1\} \end{aligned}$$

so that (5) holds whenever $y \in L_{n\alpha}$. Using (7) we see that $g_n(\alpha' t) \le g_n(t)$ whenever $L_{n\alpha'} \subseteq L_{n1}$, and since this inclusion holds for all $\alpha' \ge 1$ we conclude that g_n is nonincreasing.

To complete the proof we must establish (4). From (6) we infer the existence of a sequence $\{y_m\}$ from L_{n1} such that

(8) $$g_n(m) \le 2y_m(m), \qquad \|y_m\| = 1, \qquad m = 1, 2, \cdots.$$

Defining

(9) $$u_m(t) = y_m(mt), \qquad t \ge 0,\ m = 1, 2, \cdots,$$

we see that

(10) $$u_m \in L_{nm}, \qquad \|u_m\| = 1, \qquad m = 1, 2, \cdots.$$

In [1, Lemma 2] it is shown that a sequence $\{u_m\}$ with the properties (10) converges uniformly to zero on compact subsets of $(0, +\infty)$ and using this together with (8)–(9) we see that as $m \to \infty$

$$\overline{\lim}\, g_n(m) \le 2\, \overline{\lim}\, y_m(m) = 2\, \overline{\lim}\, u_m(1) = 0$$

from which (4) follows since g_n is nonincreasing. □

Since the function $y(t) = \exp(-t)$ is contained in L_{n1} we have $g_n(0) = 1$ and by using (4) we see that

(11) $$T_n = \sup\{t \ge 0\colon g_n(t) = 1\}$$

is finite for each $n = 1, 2, \cdots$. This being the case each $y \in L_{n\alpha}$ takes its maximum modulus somewhere in a finite interval which depends only on n and α.

Corollary 1. *Let* $y \in L_{n\alpha}$ *with* $\|y\| > 0$. *Then*

$$\|y\| = \max\{|y(t)|: 0 \leq t \leq T_n/\alpha\} \tag{12}$$

with

$$|y(t)| < \|y\| \tag{13}$$

for all $t > T_n/\alpha$.

Proof. By using (11) and (5) we obtain (12) and (13). □

Corollary 2. *Let* A *be an* $n \times n$ *matrix of complex numbers with eigenvalues* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *which satisfy* (2), *let* $\mathbf{y}: [0, +\infty) \to \mathbf{C}^n$ *satisfy the differential equation*

$$D\mathbf{y}(t) = A\mathbf{y}(t), \qquad t \geq 0,\ D = d/dt, \tag{14}$$

and let $\|\mathbf{y}\| = \max\{|\mathbf{y}(t)|_\infty: t \geq 0\}$ *where*

$$|\mathbf{y}(t)|_\infty = \max\{|y_1(t)|, |y_2(t)|, \cdots, |y_n(t)|\}$$

and where $y_1, y_2, \cdots, y_n$ *are the components of* $\mathbf{y}$. *Then*

$$|\mathbf{y}(t)|_\infty \leq \|\mathbf{y}\| g_n(\alpha t), \qquad t \geq 0. \tag{15}$$

Proof. Using (14) together with the Cayley-Hamilton theorem we find

$$[(D-\lambda_1)(D-\lambda_2)\cdots(D-\lambda_n)]\mathbf{y}(t) = [(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)]\mathbf{y}(t) = \mathbf{0}$$

so that each component of $\mathbf{y}$ satisfies (1) and (2). This being the case (15) follows at once from (5). □

3. The special case of real exponential parameters. We define $\Lambda_{n\alpha}$ to be the set of those real valued functions $y \in L_{n\alpha}$ which satisfy some differential equation (1) in which each λ_i is real, and we let

$$\gamma_n(t) = \sup\{|y(t)|: y \in \Lambda_{n1}, \|y\| = 1\}, \qquad t \geq 0, \tag{16}$$

denote the corresponding envelope function so that Theorem 1 and its corollaries hold when $L_{n\alpha}$, g_n are replaced by $\Lambda_{n\alpha}$, γ_n, respectively. We shall present three lemmas which lead to a useful alternative representation for γ_n.

Lemma 1. *Given* $n = 1, 2, \cdots$ *and* $\tau \geq 0$ *there exists some* $y \in \Lambda_{n1}$

with $\|y\| = 1$ *such that* $y(\tau) = \gamma_n(\tau)$, *i.e., the supremum in* (16) *is actually a maximum.*

Prooof. The exponential function $y(t) = \exp(-t)$ satisfies the lemma when $\tau = 0$. In the remainder of the proof we will assume that $\tau > 0$. From (16) we infer the existence of a normalized sequence $\{y_m\}$ from Λ_{n1} such that $\{y_m(\tau)\}$ converges to $\gamma_n(\tau)$. In [1, Lemmas 1, 2] it is shown that from such a normalized sequence from Λ_{n1} we may extract a subsequence which converges uniformly on compact subsets of $(0, +\infty)$ to some $y \in \Lambda_{n1}$ with $\|y\| \le 1$. For this choice of y we have

$$\gamma_n(\tau) = \lim y_m(\tau) = y(\tau) \le \|y\| \cdot \gamma_n(\tau) \le \gamma_n(\tau)$$

so that $\|y\| = 1$ and $y(\tau) = \gamma_n(\tau)$. □

As the pointwise supremum of a family of continuous real valued functions, γ_n is lower semicontinuous. Using Lemma 1 we can show that γ_n is also upper semicontinuous and thus continuous. (Indeed, let $\{t_m\}$ be any sequence of nonnegative real numbers with limit $t > 0$ and for each $m = 1, 2, \cdots$ let $y_m \in \Lambda_{n1}$ be chosen such that $y_m(t_m) = \gamma_n(t_m)$ and $\|y_m\| = 1$. After passing to a subsequence, if necessary, we may assume that $\{y_m\}$ converges uniformly on compact subsets of $(0, +\infty)$ to some $y \in \Lambda_{n1}$ with $\|y\| \le 1$, cf. [1, Lemmas 1, 2], so that

$$\overline{\lim}\, \gamma_n(t_m) = \overline{\lim}\, y_m(t_m) = \overline{\lim}\, y(t_m) = y(t) \le \gamma_n(t).)$$

We conjecture that Lemma 1 also holds in the case where Λ_{n1}, γ_n are replaced by L_{n1}, g_n, respectively, and that the (lower semicontinuous) envelope function g_n is likewise continuous.

Lemma 2. *Let* $y \in \Lambda_{n1}$ *with* $\|y\| = 1$ *and assume that*

$$y(\tau) = \gamma_n(\tau) < 1 \tag{17}$$

for some $\tau > 0$. *Then* y *may be written in the form*

$$y(t) = q(t) \cdot \exp(-t), \qquad t \ge 0, \tag{18}$$

where q *is a polynomial of degree* $n - 1$.

Proof. Since $y \in \Lambda_{n1}$ we may parametrize y in the form

$$y(t) = Y(\mathbf{c}, \boldsymbol{\lambda}, t) = \sum_{i=1}^{l} \sum_{j=1}^{k_i} c_{ij} t^{j-1} \exp(\lambda_i t) \tag{19}$$

where $\lambda_1 < \lambda_2 < \cdots < \lambda_l \le -1$ with $1 \le l \le n$, where $k_1, k_2, \cdots, k_l$ are positive integers with sum n, and where $\mathbf{c}$, $\boldsymbol{\lambda}$ denote the n coefficients c_{ij} and the l exponential parameters λ_i, respectively.

Let $z_1 < z_2 < \cdots < z_m$ be the zeros of y in $(0, \tau)$ where y changes algebraic sign (if any such zeros exist). We shall show that $m = n - 1$. Indeed if $m < n - 1$, then we can find coefficients $\mathbf{c}^* \neq \mathbf{0}$ so that the function

$$h(t) = Y(\mathbf{c}^*, \boldsymbol{\lambda}, t), \qquad t \ge 0, \tag{20}$$

will have $n - 1$ zeros with m of these zeros located at the points $z_1, z_2, \cdots, z_m$ and the remaining $n - 1 - m$ located near τ in such a manner that

$$\begin{aligned} &\operatorname{sgn} h(t) = -\operatorname{sgn} y(t) \quad \text{whenever } y_n(t) \ge [1 + y_n(\tau)]/2, \\ &\operatorname{sgn} h(\tau) = \operatorname{sgn} y(\tau) \end{aligned} \tag{21}$$

(where

$$\operatorname{sgn} x = \begin{cases} 0 & \text{if } x = 0, \\ x/|x| & \text{if } x \neq 0, \end{cases}$$

is the signum function). This will always be possible since the set of basis functions

$$\phi_{ij}(t) = t^{j-1}\exp(\lambda_i t), \qquad j = 1, 2, \cdots, k_i;\ i = 1, 2, \cdots, l, \tag{22}$$

appearing in (19) forms a Haar system of order n, cf. [3, p. 177]. But if (17), (20), (21) hold, then for all sufficiently small $\epsilon > 0$ the function

$$y_\epsilon(t) = Y(\mathbf{c} + \epsilon\mathbf{c}^*, \boldsymbol{\lambda}, t) = y(t) + \epsilon h(t)$$

will simultaneously have the properties

$$y_\epsilon \in \Lambda_{n1} \quad \text{with } \|y_\epsilon\| < 1, \tag{23}$$

$$y_\epsilon(\tau) > y_n(\tau) \tag{24}$$

in contradiction to (16). Thus we conclude $m = n - 1$ and that $c_{ik_i} \neq 0$ for each $i = 1, 2, \cdots, l$ (since the system (22) which remains when ϕ_{ik_i} is removed is a Haar system of order $n - 1$).

Under these circumstances we can find parameters $\mathbf{c}^*$ and $\boldsymbol{\lambda}^* = (\lambda_1^*, 0, 0, \cdots, 0)$ such that if

(25) $$h(t) = Y(\mathbf{c}^*, \lambda, t) + \lambda_1^* \sum_{j=1}^{k_1} c_{1j} t^j \exp(\lambda_1 t)$$

then (21) holds. Again this is possible because (25) represents an arbitrary linear combination of the n basis functions ϕ_{ij} of (22) and (since $c_{1k_1} \neq 0$) the corresponding function

$$\phi_{1,k_1+1}(t) = t^{k_1} \exp(\lambda_1 t)$$

which together form a Haar system of order $n + 1$. This being the case we see that if $\lambda_1 < -1$ then for all sufficiently small $\epsilon > 0$ the function

$$y_\epsilon(t) = Y(\mathbf{c} + \epsilon\mathbf{c}^*, \lambda + \epsilon\lambda^*, t) = y(t) + \epsilon h(t) + o(\epsilon)$$

will again simultaneously have the properties (23) and (24) which is impossible. Hence $\lambda_1 = -1$ and we conclude that y has the form (18). □

We shall now let

(26) $$q_n(t) = a_{n0} + a_{n1}t + \cdots + a_{nn}t^n, \qquad n = 0, 1, \cdots,$$

denote the Chebyshev polynomial of degree n with respect to the semi-infinite interval $[0, +\infty)$ and the weight function $w(t) = \exp(-t)$, i.e., q_n is selected so that the leading coefficient a_{nn} in (26) is maximized subject to the constraint that $\max\{|q_n(t)| \exp(-t): t \geq 0\} \leq 1$, cf. [2]. Using slight variations of standard arguments (cf. [4, pp. 34–52]) it can be shown that this condition uniquely determines q_n and that there is a unique set of extreme points $0 = \tau_{n0} < \tau_{n1} < \cdots < \tau_{nn}$ such that

(27) $$q_n(\tau_{ni}) \exp(-\tau_{ni}) = (-1)^{n-i}, \qquad i = 0, 1, \cdots, n.$$

Lemma 3. *The envelope functions defined by* (16) *can be written in the form*

(28) $$\gamma_{n+1}(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \tau_{nn}, \\ q_n(t)\exp(-t) & \text{if } t \geq \tau_{nn}, \end{cases} \qquad n = 0, 1, \cdots.$$

Proof. Let $y_n(t) = q_n(t)\exp(-t)$, $t \geq 0$, so that by construction $y_n \in \Lambda_{n+1,1}$ with $\|y_n\| = 1$ and (using (27))

(29) $$y_n(\tau_{ni}) = (-1)^{n-i}, \qquad i = 0, 1, \cdots, n.$$

From (16) and (29) we see that $\gamma_{n+1}(\tau_{nn}) = 1$, and since γ_{n+1} is nonincreasing we obtain (28) for the case where $t \leq \tau_{nn}$.

Suppose now that $\tau_{n,n+1} > \tau_{nn}$ is chosen so large that $\gamma_{n+1}(\tau_{n,n+1}) < 1$. Using Lemma 2 we infer the existence of some polynomial q of degree n such that if $y(t) = q(t)\exp(-t)$ then $y \in \Lambda_{n+1,1}$ with $\|y\| = 1$ and

$$y(\tau_{n,n+1}) = \gamma_n(\tau_{n,n+1}) \geq y_n(\tau_{n,n+1}). \tag{30}$$

Using (29), (30) and the normalization conditions $\|y\| = \|y_n\| = 1$ we find

$$(-1)^{n-i}[y(\tau_{ni}) - y_n(\tau_{ni})] \leq 0, \qquad i = 0, 1, \cdots, n+1,$$

and by removing the exponential factors we have

$$(-1)^{n-i}[q(\tau_{ni}) - q_n(\tau_{ni})] \leq 0, \qquad i = 0, 1, \cdots, n+1,$$

i.e. the polynomial $q - q_n$ of degree at most n is alternately nonnegative and nonpositive on the $n+2$ distinct points τ_{ni}. It follows that $q - q_n \equiv 0$ and thus that (28) holds whenever $t > \tau_{nn}$ and $\gamma_{n+1}(t) < 1$. But since γ_{n+1} is continuous and since $y_n(t) < 1$ for all $t > \tau_{nn}$, it follows that (28) holds for all $t \geq \tau_{nn}$. □

The results of this section may be summarized as follows.

Theorem 2. *Let $y \in \Lambda_{n\alpha}$. Then*

$$|y(t)| \leq \|y\|\gamma_n(\alpha t), \qquad t \geq 0, \tag{31}$$

where γ_n is given by (28) *and where*

$$\|y\| = \max\{|y(t)|: 0 \leq t \leq \tau_{n-1,n-1}\}. \tag{32}$$

Moreover, the bound (31) *is the best possible.*

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