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DIFFERENTIABILITY OF THE EXPONENTIAL OF A MEMBER OF A NEAR-RING

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ABSTRACT. Suppose S is a Banach space and K is the near-ring of all zero preserving Lipschitz transformations from S to S. It is shown that all exponentials of members of K have certain differentiability properties. This leads to the fact that no neighborhood of the identity transformation is filled with exponentials of members of K.

It is known ([1], [3] or [2]) that if K is a Banach algebra with identity I and T is an element of K such that $|T - I| < 1$, then $T = \operatorname{Exp} A$ for some $A \in K$. Here $\operatorname{Exp} A = \lim_{n\to\infty}(I + (1/n)A)^n$. From this fact it follows easily that the identity component of K is precisely the set of all finite products of exponentials of members of K.

One purpose of this note is to show that in a near-ring of zero preserving Lipschitz transformations on a Banach space, it no longer holds that all elements in some neighborhood of I are exponentials. It was shown in [2] that for such near-rings the finite products of exponentials are dense in the identity component. The results of this note indicate that [2] cannot be improved in a certain direction.

Denote by S a real Banach space and by K a near-ring of zero preserving Lipschitz transformations from S to S. Precisely, K is a collection of transformations from S to S such that:

(1) if each of T and $V \in K$ and c is a number, then $T + V$, TV and $cT \in K$;

(2) if $T \in K$, there is a smallest number $|T|$ (called the norm of T) such that $\|Tx - Ty\| \leq |T|\,\|x - y\|$ for all $x, y \in S$;

(3) $T0 = 0$;

(4) I, the identity transformation on S, is in K;

(5) K is complete in the sense that if T is a transformation from S to S such that (2) holds and $\{T_i\}_{i=1}^{\infty}$ is a sequence of elements of K which converges to T uniformly on each bounded subset of S, then $T \in K$.

If $A \in K$ then $\operatorname{Exp} A$ denotes the element L of K such that $Lx =$

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$\lim_{n\to\infty}(I + (1/n)A)^n x$ for all $x \in S$. Properties of this exponential function are listed in [2].

Theorem 1. *Suppose each of A and B is in K and* $(\operatorname{Exp} tA)(\operatorname{Exp} sB) = (\operatorname{Exp} sB)(\operatorname{Exp} tA)$ *for all numbers t and s. If $x \in S$, t is a number and $Bx \neq 0$, then* $\operatorname{Exp} tA$ *is G-differentiable at x in the direction Bx and*

$$(D_{Bx}\operatorname{Exp} tA)x = (B\operatorname{Exp} tA)x \quad \textit{for all numbers } t.$$

Proof. From [2, Lemma 0 (iv)], it follows that if $Q \in K$ and $x \in S$, then

$$\|(\operatorname{Exp} \delta Q)x - (I + \delta Q)x\| = \|(\operatorname{Exp} \delta Q)x - (x + \delta Qx)\| = o(\delta) \quad \text{as } \delta \longrightarrow 0.$$

Therefore for $x \in S$ and $Bx \neq 0$,

$$\begin{aligned}
&\|[(\operatorname{Exp} tA)(x + \delta Bx) - (\operatorname{Exp} tA)x] - \delta(B\operatorname{Exp} tA)x\| \\
&\quad = \|(I + \delta B)(\operatorname{Exp} tA)x - (\operatorname{Exp} tA)(I + \delta B)x\| \\
&\quad \leq \|(I + \delta B)(\operatorname{Exp} tA)x - (\operatorname{Exp} \delta B)(\operatorname{Exp} tA)x\| \\
&\qquad + \|(\operatorname{Exp} tA)(\operatorname{Exp} \delta B)x - (\operatorname{Exp} tA)(I + \delta B)x\| \\
&\quad \leq \|(\operatorname{Exp} \delta B)(\operatorname{Exp} tA)x - (I + \delta B)(\operatorname{Exp} tA)x\| \\
&\qquad + e^{|tA|}\|(\operatorname{Exp} \delta B)x - (I + \delta B)x\| \\
&\quad = o(\delta) \quad \text{as } \delta \longrightarrow 0.
\end{aligned}$$

So, $\|[(\operatorname{Exp} tA)(x + \delta Bx) - (\operatorname{Exp} tA)x] - \delta(B\operatorname{Exp} tA)x\| = o(\delta)$ as $\delta \longrightarrow 0$. But this is precisely the statement that $\operatorname{Exp} tA$ is G-differentiable at x in the direction Bx and that $(D_{Bx}\operatorname{Exp} tA)x = (B\operatorname{Exp} tA)x$.

Corollary. *If $A \in K$, t is a number and $x \in S$, then* $(D_{Ax}\operatorname{Exp} tA)x = (A\operatorname{Exp} tA)x$.

This follows from Theorem 1 since

$$(\operatorname{Exp} tA)(\operatorname{Exp} sA) = \operatorname{Exp}(t + s)A = (\operatorname{Exp} sA)(\operatorname{Exp} tA)$$

for all numbers t and s.

Theorem 2. *Suppose K is the near-ring of all zero preserving Lipschitz transformations on S. Then no neighborhood of I is filled with exponentials of members of K.*

Proof. Suppose $\epsilon > 0$. Pick $y_\epsilon \in S$ such that $\|y_\epsilon\| = \epsilon$ and define

$$T_\epsilon z = \begin{cases} z & \text{if } \|z - y_\epsilon\| \geq \epsilon, \\ z - (\epsilon - \|z - y_\epsilon\|)y_\epsilon & \text{if } \|z - y_\epsilon\| \leq \epsilon. \end{cases}$$

It may be verified that $|T_\epsilon - I| = \epsilon$, and if $x \in S$, $\|x\| \neq 0$, α is a number different from 0 so that $|\alpha| < \epsilon/\|x\|$, then

$$(1/\alpha)[(T_\epsilon - I)(y_\epsilon + \alpha x) - (T_\epsilon - I)y_\epsilon] = (|\alpha|/\alpha)\|x\|y_\epsilon.$$

Since $\lim_{\alpha \to 0}(|\alpha|/\alpha)y_\epsilon$ does not exist, it follows that $T_\epsilon - I$ (and hence T_ϵ) is not G-differentiable in any direction at y_ϵ.

Suppose now that $T_\epsilon = \operatorname{Exp} A$ for some $A \in K$. Then $Ay_\epsilon \neq 0$, since if it were zero, then

$$(1 - \epsilon)y_\epsilon = T_\epsilon y_\epsilon = (\operatorname{Exp} A)y_\epsilon = \lim_{n \to \infty} (I + (1/n)A)^n y_\epsilon = y_\epsilon,$$

a contradiction.

By the Corollary, $T_\epsilon = \operatorname{Exp} A$ is G-differentiable in the direction Ay_ϵ, a contradiction to the fact established above that T_ϵ is not G-differentiable at y_ϵ in any direction. Hence the assumption that $T_\epsilon = \operatorname{Exp} A$ for some $A \in K$ is false and the theorem is established.

The problem is left open of characterizing those near-rings of zero preserving Lipschitz transformations on S which do have a neighborhood of the identity filled with exponentials of members of the near-ring. This is a special case of the following more general problem:

Denote by K the near-ring of all zero preserving Lipschitz transformations on a Banach space S. Denote by K' the identity component of K. Characterize those subgroups of K' which have a neighborhood of I which is filled with exponentials of members of K.

A closely related problem is that of finding a necessary and sufficient condition for an element T of K ($|T - I| < 1$) to be an exponential of some member of K.

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