A CANONICAL TRANSFORMATION
NEAR A BOUNDARY POINT
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ABSTRACT. A local homogeneous canonical transformation is
constructed which straightens a curved boundary and freezes the
coefficients of the principal part of a pseudo-differential operator
in the neighborhood of a nonglancing ray.

Duistermaat and Hörmander [1] have studied the propagation along
bicharacteristics of wave front sets of solutions of certain partial differ-
etial equations, using Fourier integral operators to effect a canonical
transformation taking the given operator (locally) into $\partial / \partial x_1$. Hörmander
[2] has also studied the problem with the aid of specially constructed
pseudo-differential operators. Lax and Nirenberg [3] have applied the latter
method to the study of boundary value problems, but thus far their approach
has not handled the glancing ray case. As a first step towards adapting the approach
of [1] to deal with boundary value problems, we construct a canonical
transformation, away from glancing rays, which simultaneously reduces the
boundary and the equation to a convenient form. I wish to thank Ralph
Phillips for many helpful conversations.

Let $p(x, t; \xi, \tau)$ be a real symbol which is positive homogeneous of
degree $m \geq 0$, $m$ an integer, and with $(x, t, \xi, \tau) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$.
Let $0 \neq (\xi^0, \tau^0)$ satisfy $\partial p(0, 0; \xi^0, \tau^0) / \partial \tau \neq 0$. Let $\Gamma$ be a smooth sur-
face in $\mathbb{R}^n$, passing through $(0, 0)$, and such that the normal to $\Gamma$ at $(0, 0)$
points in the direction of the $t$ axis.

Theorem. There is a canonical map $\chi: (x, t, \xi, \tau) \mapsto (y, s, \eta, \sigma) \in \mathbb{R}^{2n}$,
defined in a conic neighborhood $U$ of $(0, 0, \xi^0, \tau^0)$, homogeneous of degree
one in $(\xi, \tau)$, and such that for $(x, t, \xi, \tau) \in U$,
(i) $(x, t) \in \Gamma \Rightarrow s = 0$,
(ii) $p(x, t; \xi, \tau) = p(0, 0; \eta, \sigma) \overset{\text{def}}{=} p_0(\eta, \sigma)$.

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Proof. For $\chi$ to be canonical means that the Poisson brackets of the image points satisfy

$$\{y_i, y_j\} = \{y_i, s\} = \{y_i, \sigma\} = \{s, \eta_i\} = \{\eta_i, \sigma\} = 0,$$

where

$$\{u, v\} = \sum_l \left( \frac{\partial u}{\partial x_l} \frac{\partial v}{\partial x_l} - \frac{\partial u}{\partial \xi_l} \frac{\partial v}{\partial \xi_l} \right) + \left( \frac{\partial u}{\partial t} \frac{\partial v}{\partial \tau} - \frac{\partial u}{\partial \tau} \frac{\partial v}{\partial t} \right),$$

and

$$H_u \stackrel{\text{def}}{=} H_v u \stackrel{\text{def}}{=} \left[ b_v^{(1)}, \left( \nabla_x, \frac{\partial}{\partial t} \right) - b_v^{(2)}, \left( \nabla_\xi, \frac{\partial}{\partial \tau} \right) \right] u, \quad b_v = (b_v^{(1)}, b_v^{(2)}).

Our construction of $\chi$ is a modification of that given by Duistermaat and Hörmander [1] in free space. The functions $\eta_i$ will be constructed successively on $Y \times R^n$ by assigning each on an initial manifold transverse to the linear span of those $b_{\eta_i}$ which are already known, and such that $b_{\eta_i}^{(1)}$ is tangential to $\Gamma$. This last fact will enable us to construct $s$ such that $s = 0$ on $\Gamma$. Once $\eta$ is constructed, $\sigma$ is determined by (ii). To extend $\eta$ and $\sigma$ off of $\Gamma$, we shall use the equation $b_p \eta = 0$ together with (ii); a simple application of the chain rule shows that this construction implies the canonical relations $\{\eta_i, \sigma\} = 0$. Finally, we shall use the initial condition $(y, s)(0, 0, \eta, \sigma) = (y, s)$ together with the equations $b_{\eta_i}(y, s) = b_{\sigma}(y, s) = 0$ to determine $(y, s)$.

We now construct $\chi$. Let $N_1$ be a neighborhood of $(0, 0)$ in $\Gamma$, and $C_1$ a conic neighborhood of $(\xi^0, r^0)$ in $R^n$ such that for $(x, t) \in N_1$ and $0 \neq (\xi, r) \in C_1$,

$$\langle n, \nabla_\xi, r \rangle \neq 0,$$

where $n$ is the normal to $\Gamma$ at $(x, t)$. On $N_1$, let $v_1(x, t)$ be a nonsingular tangential vector field such that $\langle (\xi^0, r^0), v_1(0, 0) \rangle = \xi^0_1$, and define

$$\eta_1(x, t, \xi, r) = \langle (\xi, r), v_1(x, t) \rangle, \quad (x, t, \xi, r) \in N_1 \times C_1.$$

Because of (1), $b_p^{(1)}$ is not tangential to $N_1$, and hence we can extend the definition of $\eta_1$ off of $N_1 \times C_1$ by using (2) as an initial condition for

$$H_p \eta_1 = 0.$$

We define recursively triples $\{N_i, v_i, \eta_i\}, i = 2, \cdots, n - 1$ as follows.
Let $N_i \ni (0, 0)$ be a smooth $(n-1)$-dimensional surface in $N_{i-1}$, transverse to the span of the vectors $v_1, \ldots, v_{i-1}$ and not orthogonal to $(\xi^0, r^0)$ unless $\xi^0_i = \xi^0_{i+1} = \cdots = \xi^0_{n-1} = 0$. Let $v_i$ be a nonsingular vector field in $N_i$ such that $\langle (\xi^0, r^0), v_i(0, 0) \rangle = \xi^0_i$. Define

\begin{equation}
\eta_i = \langle (\xi, r), v_i(x, t) \rangle, \quad (x, t, \xi, r) \in N_i \times C_1,
\end{equation}

and extend $\eta_i$ by the equations:

\begin{equation}
\{\eta_i, \eta_j\} = 0, \quad j < i,
\end{equation}

and

\begin{equation}
\{\eta_i, p\} = 0.
\end{equation}

The consistency of the construction of $\eta$ using (5) and (6) follows from the identity $[H_{u^i}, H_{u^j}] = H_{[u^i, u^j]}$. For example, the equations $\{\eta_j, \eta_k\} = 0$ are satisfied by construction along a submanifold $M_k$, $j < k$, and $\{\eta_j, p\} = 0$ along integral curves of $H_p$ through $M_k$. On these integral curves, then,

\begin{equation}
\{p, \{\eta_j, \eta_k\}\} = H\{\eta_j, \eta_k\}p = [H_{\eta_j}, H_{\eta_k}]p = H_{\eta_j}p, \{\eta_k, p\} - H_{\eta_k}p, \{\eta_j, p\} = 0,
\end{equation}

so that $\{\eta_j, \eta_k\} = 0$ along these curves.

Remark. If $\Gamma$ is the hyperplane $t = 0$, it suffices to set $\eta = \xi$ on $N_1 \times C_1$ and use (6) to extend the definition of $\eta_t$.

The condition $\partial p(0, 0; \xi^0, r^0)/\partial r \neq 0$, together with the above construction, ensures that there is a conic neighborhood $\mathcal{U}$ of $(0, 0; \xi^0, r^0)$ in which $\sigma$ is uniquely defined by (ii) if we set $\sigma(0, 0; \xi^0, r^0) = r^0$. Locally, $\sigma$ is defined as a function of $Z = (\eta, p)$, from which we conclude that

\begin{equation}
\{\sigma, \eta_j\} = H_{\eta_j} \sigma = \sum_k \frac{\partial \sigma}{\partial \eta_k} \{\eta_j, \eta_k\} + \frac{\partial \sigma}{\partial p} \{\eta_j, p\} = 0.
\end{equation}

According to [1], we can now determine $(y, s)$ in $\mathcal{U}$ by assigning $(y, s)$ on an $n$-dimensional manifold transverse to the span of $b_{\eta_j}$, $j = 1, \ldots, n-1$, and $b_t$, provided that these vectors together with the radial vector $(0, 0; \xi, r)$ are linearly independent. Such a manifold is the subspace $x = 0, t = 0$. To see this, we need only observe that

\begin{equation}
b_{\eta_j}(0, 0, \xi, r) = (e_j, 0, 0, \partial \eta_j/\partial t),
\end{equation}

where $e_j$ is a standard unit basis vector in $R^{n-1}$, and that the $n$th component of $b_t(0, 0, \xi, r)$ is (cf. (ii)).
We assign initial conditions

\[(y, s)(0, 0, \xi, r) = (0, 0).\]

Using (7) together with equations \(H_{\eta_i}(y, s) = (e_i 0), H_{\eta_i}(y, s) = (0, 1),\) serves to define \((y, s)\) in \(\mathcal{U}\).

There remains to show that (i) holds. But \(s\) is invariant on the integral curves of each \(H_{\eta_j}\), and if \((x, t) \in N_j\), then \(h^{(1)}_{\eta_j} = v_j\) is tangent to \(\Gamma\).

Given \((x', t', \xi', r') \in \mathcal{U}\) with \((x', t') \in N_1\), we follow successively the integral curves of \(H_{\eta_i}, i = 1, \ldots, n - 1,\) through \((x^i, t^i, \xi^i, r^i)\) till \((x, t)\) hits \(N_{i+1}\) at \((x^{i+1}, t^{i+1})\) and \((\xi^i, r) = (\xi^{i+1}, r^{i+1})\), with \(N_n\) defined as the point \((0, 0)\). We conclude that for some \(P = (0, 0, \xi^n, r^n) \in \mathcal{U}\), \(s(x, t, \xi, r) = s(P) = 0\) by (7). Theorem 1 is proved.

As a corollary of the proof, we note that if \(\Gamma\) is the hyperplane \(t = 0\), then \(\chi\) can be extended to a conical neighborhood of any cone \(C = (0, 0, V \setminus \{0\})\), where \(V\) is a closed simply connected cone, and where \(\partial p/\partial r \neq 0\) on \(C \setminus \{0\}\). Since \(\chi(0, 0, \xi, r) = (0, 0, \xi, r)\), condition (ii), together with a simple homotopy argument, allows us to drop the assumption that \(V\) be simply connected.

Remark. If \(p\) has the parity of \(m\), then \(\chi\) extends by homogeneity to a two-sided conic set.

Some applications and extensions will be reported on elsewhere.

REFERENCES

