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ON CONSECUTIVE INTEGER SOLUTIONS FOR $y^2 - k = x^3$

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ABSTRACT. In this paper the number of k's having a given number of consecutive integer solutions either for x or for y or for both in the equation $y^2 - k = x^3$ has been found.

Introduction. The equation $y^2 - k = x^3$, now known as Mordell's equation, has interested mathematicians for more than three centuries, and has played an important role in the development of number theory. Many interesting, important, and beautiful results on this subject have been published. For a complete bibliography one can see [1]. It is really surprising to see that an interesting problem like finding the number of k's having a given number of solutions of a given type has been overlooked by all for such a long time.

In a recent paper [2] we have given the identity

$$(t^3 - 3)^2 - 2^3 = (t^3 + 1)^2 - (2t)^3 = (3t^3 - 1)^2 - (2t^2)^3 = t^6 - 6t^3 + 1.$$

For even t this shows that there is an infinite number of k's having six solutions $(x, \pm y)$ where x and y are coprime. This result also follows from the identity

$$(4a^3 + 1)^2 - (2a)^3 = (4a^3)^2 - (-1)^3 = (4a^3 - 1)^2 - (2a^3 = 16a^6 + 1,$$

which is obtained in this paper.

We want to find the number of k's having a given number of consecutive integer solutions for y or x or both.

Theorem 1. *If $y^2 - k = x^3$ (k an integer) has two consecutive integer solutions for y, then k is a positive integer.*

Proof. If x is a negative integer, then $k = y^2 - x^3$ is always positive. So we may assume that x takes only positive integer values. Let (x_1, y_1) and $(x_2, y_1 - 1)$ be two integer solutions for $y^2 - k = x^3$. Then we have

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(1) $$y_1^2 - k = x_1^3$$

and

(2) $$(y_1 - 1)^2 - k = x_2^3.$$

From (1) and (2) we get

(3) $$y_1 = (x_1^3 - x_2^3 + 1)/2.$$

We note that $x_1 > x_2$ or $x_2 > x_1$ accordingly as y_1 is positive or negative. Now we have

(4) $$k = ((x_1^3 - x_2^3 + 1)/2)^2 - x_1^3 \quad \text{when } x_1 > x_2,$$

and

(5) $$k = ((x_2^3 - x_1^3 + 1)/2)^2 - x_2^3 \quad \text{when } x_2 > x_1.$$

It is sufficient to prove that $k = ((x_1^3 - x_2^3 + 1)/2)^2 - x_1^3$ is a positive integer. Since $x_1 > x_2$, we can write $x_2 \leq x_1 - 1$. Then we have

$$((x_1^3 - x_2^3 + 1)/2)^2 \geq ((3x_1^2 - 3x_1 + 2)/2)^2,$$

for $(3x_1^2 - 3x_1 + 2)/2$ is always a positive integer. Therefore, we have

$$k = \left(\frac{x_1^3 - x_2^3 + 1}{2}\right)^2 - x_1^3 \geq \left(\frac{3x_1^2 - 3x_1 + 2}{2}\right)^2 - x_1^3.$$

But

$$\left(\frac{3x_1^2 - 3x_1 + 2}{2}\right)^2 - x_1^3 = \frac{1}{4}(x_1 - 1)^2(x_1 - 2)^2 + 2x_1^2(x_1 - 1)^2$$

is a positive integer. Hence k is a positive integer.

Theorem 2. *There is no k for which $y^2 - k = x^3$ has five consecutive integer solutions for y.*

Proof. Suppose that there exists a k for which $y^2 - k = x^3$ has solutions given by (x_1, y_1), $(x_2, y_1 - 1)$, $(x_3, y_1 - 2)$, $(x_4, y_1 - 3)$ and $(x_5, y_1 - 4)$. Then we must have

$$x_1^3 - 2x_2^3 + x_3^3 = x_2^3 - 2x_3^3 + x_4^3 = x_3^3 - 2x_4^3 + x_5^3 = 2.$$

Since the cube of an integer is congruent to 0, 1 or -1 modulo 9, $x_1^3 - 2x_2^3 + x_3^3 \equiv 2 \pmod 9$ is possible only in the following cases.

(i) $x_1^3 \equiv 1$, $x_2^3 \equiv 0$, $x_3^3 \equiv 1$;

(ii) $x_1^3 \equiv 0$, $x_2^3 \equiv -1$, $x_3^3 \equiv 0$;
(iii) $x_1^3 \equiv 1$, $x_2^3 \equiv -1$, $x_3^3 \equiv -1$;
(iv) $x_1^3 \equiv -1$, $x_2^3 \equiv -1$, $x_3^3 \equiv 1$.

We have also $x_2^3 - 2x_3^3 + x_4^3 \equiv 2 \pmod 9$. Then we get $x_4^3 \equiv 4, 3, 1$ or $5 \pmod 9$ accordingly as (i), (ii), (iii), or (iv) holds. The two congruences, $x_1^3 - 2x_2^3 + x_3^3 \equiv 2$ and $x_2^3 - 2x_3^3 + x_4^3 \equiv 2$, are simultaneously satisfied only for $x_1^3 \equiv 1$, $x_2^3 \equiv -1$, $x_3^3 \equiv -1$ and $x_4^3 \equiv 1$ all modulo 9. But then $x_3^3 - 2x_4^3 + x_5^3 \equiv 5, 6$ or $7 \pmod 9$, contradicting the fact that $x_3^3 - 2x_4^3 + x_5^3 \equiv 2 \pmod 9$.

Hence we have proved that there is no k for which $y^2 - k = x^3$ has five consecutive integer solutions for y.

Corollary. *If* (x_1, y_1), $(x_2, y_1 - 1)$, $(x_3, y_1 - 2)$ *and* $(x_4, y_1 - 3)$ *are solutions for* $y^2 - k = x^3$, *then* $y_1 \equiv 33 \pmod{63}$ *and* $k \equiv 17 \pmod{63}$.

Proof. From Theorem 2, we see that $x_1^3 \equiv 1$, $x_2^3 \equiv -1$, $x_3^3 \equiv -1$ and $x_4^3 \equiv 1$ all modulo 9. Since the cube of an integer is again congruent to 0, 1 or -1 modulo 7, arguing as before we will again have $x_1^3 \equiv 1$, $x_2^3 \equiv -1$, $x_3^3 \equiv -1$, and $x_4^3 \equiv 1 \pmod 7$. Combining both we have $x_1^3 \equiv 1 \pmod{63}$, $x_2^3 \equiv -1 \pmod{63}$, $x_3^3 \equiv -1 \pmod{63}$, and $x_4^3 \equiv 1 \pmod{63}$.

Now from $y_1^2 - k = x_1^3$ and $(y_1 - 1)^2 - k = x_2^3$, we get $2y_1 - 1 = x_1^3 - x_2^3 \equiv 65 \pmod{126}$, whence $y_1 \equiv 33 \pmod{63}$. Then $k = y_1^2 - x_1^3 \equiv 17 \pmod{63}$.

Let $y^2 - k = x^3$ (k an integer) possess three consecutive integer solutions given by (x_1, y_1), $(x_2, y_1 - 1)$ and $(x_3, y_1 - 2)$. Then we get

$$(6) \qquad 2y_1 = x_1^3 - x_2^3 + 1 = x_2^3 - x_3^3 + 3 \quad \text{whence } x_1^3 + x_3^3 = 2(1 + x_2^3).$$

It is easy to see that x_1, x_2 and x_3 are respectively odd, even and odd or even, odd and even and, moreover, $x_1 > x_2 > x_3$ or $x_1 < x_2 < x_3$. We give below four parametric solutions, (a), (b), (c) and (d), for the equation $x_1^3 + x_3^3 = 2(1 + x_2^3)$.

(a) $x_1 = 1 + t - t^2$, $x_2 = -t^2$ and $x_3 = 1 - t - t^2$;
(b) $x_1 = 1 + 3m^3$, $x_2 = 3m^2$ and $x_3 = 1 - 3m^3$;
(c) $x_1 = 2a$, $x_2 = -1$ and $x_3 = -2a$;
(d) $x_1 = 72t^4$, $x_2 = 36t^3 - 1$ and $x_3 = 6t - 72t^4$.

In (a) x_1 and x_3 are always odd for any integer t. To make x_2 an even integer we put $t = 2r$. It is easy to see that $x_1 > x_2 > x_3$ or $x_1 < x_2 < x_3$ accordingly as r is a positive or a negative integer.

Then we have

$$(1 + 3r - 20r^3 + 48r^5)^2 - (1 + 2r - 4r^2)^3 = (3r - 20r^3 + 48r^5)^2 - (-4r^2)^3$$

(a′)
$$= (3r - 20r^3 + 48r^5 - 1)^2 - (1 - 2r - 4r^2)^3$$

$$= 2304r^{10} - 1920r^8 + 752r^6 - 120r^4 + 9r^2.$$

From (b) we get another identity:

$$\left(\frac{2 + 9r^3 + 27r^9}{2}\right)^2 - (1 + 3r^3)^3 = \left(\frac{9r^3 + 27r^9}{2}\right)^2 - (3r^2)^3$$

(b′)
$$= \left(\frac{9r^3 + 27r^9 - 2}{2}\right)^2 - (1 - 3r^3)^3 = \frac{1}{4}(729r^{18} + 486r^{12} - 27r^6).$$

For $a > 0$, (c) yields yet another identity:

(c′) $(4a^3 + 1)^2 - (2a)^3 = (4a^3)^2 - (-1)^3 = (4a^3 - 1)^2 - (-2a)^3 = 16a^6 + 1.$

Taking $t > 0$ in (d) we have $x_1 > x_2 > x_3$ and can have a fourth identity. Each of the identities (a′), (b′), (c′) and (d′) gives an infinite number of k's having three consecutive integer solutions for y in $y^2 - x^3 = k$. Our discussion above proves

Theorem 3. *There is an infinite number of k's having three consecutive integer solutions for y in $y^2 - k = x^3$.*

Let the number of k's for which $y^2 - k = x^3$ has integer solutions given by $(x_1, y_1), (x_1 - 1, y_1 - 1), \cdots, (x_1 - i + 1, y_1 - i + 1)$ be denoted by $N_i^{(x,y)}$. $N_i^{(x)}$ and $N_i^{(y)}$ have the usual meaning. We have already seen that $N_5^{(y)} = 0$ and $N_3^{(y)} = \infty$.

Theorem 4. *We have $N_2^{(x,y)} = \infty$ and hence $N_2^{(x)} = N_2^{(y)} = \infty$.*

Proof. Let $y^2 - k = x^3$ have two consecutive integer solutions given by (x_1, y_1) and $(x_1 - 1, y_1 - 1)$. Then from $y_1^2 - k = x_1^3$ and $(y_1 - 1)^2 - k = (x_1 - 1)^3$, we get

$$y_1 = 1 + \frac{3}{2}x_1(x_1 - 1),$$

and hence

$$k = \frac{1}{4}(x_1 - 1)^2(9x_1^2 - 4x_1 + 4).$$

It is easy to see that y_1 and k are both integers for integral values of x_1. Putting arbitrary values for x_1, we get an infinite number of k's having two consecutive integer solutions for $y^2 - k = x^3$.

Theorem 5. *We have $N_3^{(x)} = 0$ and hence $N_3^{(x,y)} = 0$.*

Proof. Suppose that $y^2 - k = x^3$ has integer solutions given by (x_1, y_1), $(x_1 - 1, y_2)$ and $(x_1 - 2, y_3)$. Then we get $y_1^2 - y_2^2 = 3x_1^2 - 3x_1 + 1$ and $y_2^2 - y_3^2 = 3x_1^2 - 9x_1 + 7$.

Reading modulo 3, we have $y_1^2 - y_2^2 \equiv 1 \pmod 3$ and $y_2^2 - y_3^2 \equiv 1 \pmod 3$. From $y_1^2 - y_2^2 \equiv 1 \pmod 3$, we get $y_2^2 \equiv 0 \pmod 3$. Then $y_2^2 - y_3^2 \equiv 1 \pmod 3$ implies that $y_3^2 \equiv 2 \pmod 3$ which is absurd. Hence the theorem is proved.

The problem will be completely solved if we can find $N_4^{(y)}$. In the identity (a′), the substitution $r = 1$ gives $32^2 - (-1)^3 = 31^2 - (-4)^3 = 30^2 - (-5)^3 = 1025$, while $a = 2$ in (c′) yields $33^2 - (4)^3 = 32^2 - (-1)^3 = 31^2 - (-4)^3 = 1025$. Combining, we have

$$33^2 - (4)^3 = 32^2 - (-1)^3 = 31^2 - (-4)^3 = 30^2 - (-5)^3 = 1025.$$

This shows the existence of a k having four consecutive integer solutions for y. Moreover, if any k having three consecutive integer solutions is obtainable from one of the identities (a′), (b′), (c′) and (d′), then without much difficulty, it can be proved that $k = 1025$ is the only one having four consecutive integer solutions for y. In the Corollary of Theorem 2 we have already shown that for $y^2 - k = x^3$ to have four consecutive integer solutions for y we must have $y \equiv 33 \pmod{63}$ and $k \equiv 17 \pmod{63}$. The first such k is 1025 with solutions $(x, y) = (4, 33), (-1, 32), (-4, 31)$ and $(-5, 30)$.

We conclude this paper with one conjecture and one problem.

Conjecture. There are only finite number of k's (most probably none except $k = 1025$) having four consecutive integer solutions for y in $y^2 - k = x^3$.

Problem. Does there exist a k having three consecutive integer solutions for y in $y^2 - k = x^3$ which cannot be obtained from one of the identities given in the paper?

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