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A GEOMETRIC CHARACTERIZATION OF FRÉCHET SPACES WITH THE RADON-NIKODYM PROPERTY [1]

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ABSTRACT. Let F be a locally convex Fréchet space. F is said to have the *Radon-Nikodym* property if for every positive finite measure space (Ω, Σ, μ), and every μ-continuous vector measure $m: \Sigma \to F$ of bounded variation, there exists an integrable function $f: \Omega \to F$ such that $m(S) = \int_S f(\omega)\, d\mu(\omega)$, for every $S \in \Sigma$. Maynard proved that a Banach space has the Radon-Nikodym property iff it is an s-dentable space. It is the purpose of this paper to give the following analogous characterization: A Fréchet space F has the Radon-Nikodym property iff F is s-dentable.

0. **Introduction.** In [8], Maynard obtained some equivalent geometric conditions for the average range of a vector measure in the characterization of Rieffel's Radon-Nikodym theorem [11, Main theorem, p. 466]. Based on these results, Maynard [9, Theorem 2.2] recently extended Rieffel's [12, Theorem 1] condition on the dentability of the average range to s-dentability of the average range. It was shown in [2], [7] that all of these results can be extended to locally convex Fréchet spaces; see §2. As a consequence, the geometric characterization of Fréchet spaces having the Radon-Nikodym property will be proved in §3 below.

1. **Preliminaries.** Let (Ω, Σ, μ) be a positive finite measure space, where Ω is an abstract set, Σ is a σ-algebra of subsets of Ω, and μ is a real-valued measure defined on Σ. Without loss of generality, one can assume that Σ is μ-complete. Let $\Sigma^+ \equiv \{S \in \Sigma | \mu(S) > 0\}$.

Throughout this paper, let F be a locally convex Fréchet space, and $\mathfrak{U} \equiv \{U_n\}_{n=1}^{\infty}$ be a fundamental decreasing sequence of closed absolutely

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convex subsets which forms a 0-neighborhood base for F. For $U \in \mathfrak{U}$, let P_U denote the associated continuous seminorm.

A function $f\colon \Omega \to F$ is said to be *simple* iff f is of the form $f(\omega) = \sum_{i=1}^n x_i I_{S_i}(\omega)$, $\forall\, \omega \in \Omega$, where $\{S_i\}_{i=1}^n \subseteq \Sigma$ are disjoint and $\{x_i\}_{i=1}^n \subseteq F$. A function $f\colon \Omega \to F$ is said to be *strongly measurable* iff there exists a sequence of simple functions $\{f_n\}_{n=1}^\infty$ such that $f(\omega) = \lim_{n\to\infty} f_n(\omega)$, $\forall\, \omega \in \Omega$. Let $\mathfrak{M}(\Omega, \mu; F) = \{f\colon \Omega \to F \mid f \text{ is strongly measurable}\}$. If F is a Banach space, then $\mathfrak{M}(\Omega, \mu; F)$ is precisely the Bochner measurable functions.

Let f be strongly measurable, hence Borel measurable. Thus $\forall\, U \in \mathfrak{U}$, $p_U(f)$ is Borel measurable. If $\forall\, U \in \mathfrak{U}, \int_\omega p_U(f)\, d\mu < \infty$, then f is said to be *integrable.* In this case define $q_U(f) \equiv \int_\Omega p_U(f)\, d\mu$. Let $\mathfrak{L}^1(\Omega, \mu; F)$ be the space of all integrable functions, and let $\mathfrak{L}^1(\Omega, \mu; F)/\mathfrak{N}$, where $\mathfrak{N} \equiv \{f \in \mathfrak{L}^1(\Omega, \mu; F) | q_U(f) = 0, \forall\, U \in \mathfrak{U}\}$. $L^1(\Omega, \mu; F)$ is a Fréchet space topologized by the family of seminorms $\{q_U | U \in \mathfrak{U}\}$. If F is a Banach space, then $L^1(\Omega, \mu; F)$ is just the Banach space of Bochner integrable functions.

Let $m\colon \Sigma \to F$ be a vector measure. For every $U \in \mathfrak{U}$, the *U-variation of m over S* is defined to be

$$V(m, U)(S) \equiv \sup \left\{ \sum_{i=1}^n p_U(m(S_i)) | S_i \in \Sigma, \text{ disjoint, } 1 \le i \le n, S_i \subseteq S \right\}.$$

$V(m, U)(\cdot)$ is an extended real-valued measure. m is said to have *bounded variation* if $V(m, U)(\Omega) < \infty$, $\forall\, U \in \mathfrak{U}$. m is *μ-continuous*, denoted by $m \ll \mu$ if $\mu(S) = 0$, $S \in \Sigma$, implies $m(S) = 0$. It is clear that $m \ll \mu$ iff $\forall\, U \in \mathfrak{U}$, $V(m, U)(\cdot) \ll \mu$. If $f \in L^1(\Omega, \mu; F)$ and $\mu_f(S) \equiv \int_S f\, d\mu$, then μ_f is a vector measure, and $\forall\, U \in \mathfrak{U}$, $p_U(\mu_f(S)) \le \int_S p_U(f)\, d\mu$, $V(\mu_f, U)(S) = \int_S p_U(f)\, d\mu$, and μ_f is of bounded variation.

For $S \in \Sigma^+$, *the average range of m over S* is defined to be the set

$$A_S(m) \equiv \{m(T)/\mu(T) | T \in \Sigma^+, \text{ and } T \subseteq S\}.$$

Let E be an arbitrary locally convex space, and $D \subseteq E$ be a subset. $\overline{D}$ denotes the *closure* of D, $\overline{D}^\sigma$ the $\sigma(E, E^1)$-*closure*, $c(D)$ the *convex hull* of D, and

$$s(D) \equiv \left\{ \sum_{i=1}^\infty \alpha_i d_i | \alpha_i > 0, \ \sum_{i=1}^\infty \alpha_i = 1, \text{ and } \sum_{i=1}^\infty \alpha_i d_i \text{ converges, } d_i \in D, i \ge 1 \right\},$$

the s-convex hull of D. It is clear that $D \subseteq c(D) \subseteq s(D) \subseteq \overline{c}(D)$. *$D$ is said to have width at most U* iff $D - D \subseteq U$, i.e., $p_U(x - y) < 1$, $\forall\, x, y \in D$.

Definition 1.1. $D \subset E$ is said to be *dentable* [*s-dentable*] iff $\forall\, U \in \mathfrak{U}$, there exists $d \in D$ such that $d \notin \overline{c}(D \backslash \{d + U\})$ [$d \notin s(D \backslash \{d + U\})$].

If D is not dentable [s-dentable], then any $U \in \mathfrak{U}$ such that $\forall\, d \in D$, $d \in \overline{c}(D \setminus \{d + U\})$ [$d \in s(D \setminus \{d + U\})$] is said to be a *nondenting* [*non-s-denting*] *neighborhood for* D. If D is dentable, then D is s-dentable. For the proof of the following results as well as other properties of dentability and s-dentability, see [2], [9], [10], [12].

Theorem 1.1. *Let F be a Fréchet space. Every relatively weakly compact subset of F is dentable, and hence s-dentable.*

2. **Radon-Nikodym theorems.** Theorem 2.1 and Theorem 2.2 below are proved in [2], [7]. For the Banach space version, Theorem 2.1 was due to Rieffel [11, Main theorem, p. 466], and Theorem 2.2 was due to Maynard [8, Theorem 3.1, p. 457].

Theorem 2.1. *Let (Ω, Σ, μ) be a positive finite measure space, and F a Fréchet space. Let $m\colon \Sigma \to F$ be a vector measure. Then $m = \mu_f$ for some $f \in L^1(\Omega, \mu; F)$ iff*

(i) $m \ll \mu$,

(ii) *m has bounded variation,*

(iii) *m has locally relatively compact [or s-dentable] average range, i. e., $\forall\, S \in \Sigma^+$, there exists $T \in \Sigma^+$, $T \subseteq S$ such that $A_T(m)$ is relatively compact [or s-dentable].*

The next theorem will be used crucially in the proof of Theorem 3.1.

Theorem 2.2. *Let (Ω, Σ, μ) be a positive finite measure space, F a Fréchet space, and $m\colon\Sigma \to F$ a μ-continuous vector measure of bounded variation. Then the following conditions are equivalent:*

(i) *given $S \in \Sigma^+$, there exists $T \in \Sigma^+$, $T \subseteq S$ such that $A_T(m)$ is relatively compact.*

(ii) *given $S \in \Sigma^+$, and $U \in \mathfrak{U}$, there exists $T \in \Sigma^+$, $T \subseteq S$ such that $A_T(m)$ has width at most U, i.e., $A_T(m) - A_T(m) \subseteq U$.*

3. **On Fréchet spaces with the Radon-Nikodym property.** As in the case of Banach spaces, the concept of s-dentability provides a simple characterization of Fréchet spaces with the Radon-Nikodym property.

Definition 3.1. A Fréchet space F is said to have the *Radon-Nikodym* Property, RNP for short, iff for any positive measure space (Ω, Σ, μ) and any $m\colon \Sigma \to F$, μ-continuous vector measure of bounded variation, there exists $f \in L^1(\Omega, \mu; F)$ such that $m = \mu_f$.

The following lemma will motivate the next definition as well as the main theorem. For a proof of this lemma, see [7, Corollary 3.2].

Lemma 3.1. *Let (Ω, Σ, μ) be a positive finite measure space, and m: $\Sigma \to F$ a μ-continuous vector measure of bounded variation; then m has locally bounded average range, i.e., $\forall S \in \Sigma^+$, there exists $T \in \Sigma^+$, $T \subseteq S$ such that $A_T(m)$ is bounded.*

Definition 3.2. A Fréchet space F is said to be *s-dentable* iff every bounded subset of F is s-dentable.

Now the main result will be proved. The proof will be given in several stages and contains a modification of the proof presented in [9, Theorem 3.1].

Theorem 3.1. *A Fréchet space F has the Radon-Nikodym property iff F is s-dentable.*

Proof. Necessity follows from Lemma 3.1 and Theorem 2.1. To prove sufficiency, it will be shown that one can construct a positive complete finite measure space (Ω, Σ, μ), and a μ-continuous vector measure m: $\Sigma \to F$ of bounded variation such that m has no Radon-Nikodym derivative.

Stage 1. Construction of the measurable space (Ω, Σ). Let $\Omega = [0, 1)$ with the Euclidean topology, and let $\{\pi_n\}_{n=1}$ be an increasing sequence of infinite partitions of $[0, 1)$ having the following properties:

(i) $\pi_n = \{A_z^n\}_{z \in N^n}$, where N is the set of positive integers, and $A_z^n = [a_z^n, b_z^n)$.

(ii) For every $n \in N$, and every $z \in N^n$, $b_{(z,i)}^{n+1} = a_{(z,i+1)}^{n+1}$.

(iii) For every $n \in N$, and every $z \in N^n$, $A_z^n = \bigcup_{i=1}^{\infty} A_{(z,i)}^{n+1}$, where (z, i) $N^n \times N$.

Let $\pi = \bigcup_{n=1}^{\infty} \pi_n$, and let $\mathfrak{R}_0$ be the ring generated by π. It should be noted that π is countable and is not a semiring. It is easy to see that

$$(1)\quad \mathfrak{R}_0 = \left\{ S_1 \cup S_2 \middle| S_1 = \bigcup_{i=1}^{n_1} A_{z_i}^{k_i}, S_2 = \bigcup_{i=1}^{n_2} \left[\bigcup_{j=p_i}^{\infty} A_{(z_i,j)}^{r_i+1} \right], S_1 \cap S_2 = \emptyset \right\},$$

and that $\mathfrak{R}_0$ is *not* an algebra. Let $\mathfrak{R}$ be the algebra generated by π. It should be pointed out that neither $\mathfrak{R}_0$ nor $\mathfrak{R}$ contains a base for the Euclidean topology of $[0, 1)$. However, $\mathfrak{R}_0$ does contain a base for the right half-open interval topology on $[0, 1)$, which is strictly finer than the Euclidean topology [6, p. 37]. Let $\mathfrak{D}$ be the δ-ring generated by π, $\mathfrak{B}$ the δ-ring generated by compact subsets of $[0, 1)$ ($\mathfrak{B}$ is called the family of relatively compact Borel sets of $[0, 1)$ in [5, p. 287]). Let

$$(2)\quad \mathcal{C} \equiv \{S \subseteq [0, 1) | S \cap B \in \mathfrak{B}, \forall B \in \mathfrak{B}\}.$$

$\mathcal{C}$ is a σ-algebra [5, p. 290]. One has the following inclusion relations:

$$\pi \subseteq \mathcal{R}_0 \subseteq \mathcal{D} \subseteq \mathcal{B} \subseteq \sigma(\mathcal{R}_0) = \sigma(\mathcal{R}) \subseteq \mathcal{C}. \tag{3}$$

Note that $\mathcal{R} \not\subseteq \mathcal{B}$, since $[0, 1) \notin \mathcal{B}$. The third inclusion is seen as follows. Let $[a, b) \subseteq [0, 1)$. Let $a_i = a$, $a < b_i < b$ such that $b_i < b_j$, $i < j$ and $\lim_{i\to\infty} b_i = b$. Then $[a, b) = \bigcup_{i=1}^{\infty}[a_i, b_i]$. However, $[a_i, b_i] \in \mathcal{B}$, $[a_i, b_i] \subseteq [a, b] \in \mathcal{B}$, so by property (4) of [5, p. 4], $[a, b) \in \mathcal{B}$. This implies that $\pi \subseteq \mathcal{B}$. But $\mathcal{B}$ is a δ-ring, so $\mathcal{D} \subseteq \mathcal{B}$. To see the fourth inclusion, recall that every open subset $V \subseteq [0, 1)$ has the structure

$$V = [0, b) \cup \bigcup_{i=1}^{\infty} (a_i, b_i), \tag{4}$$

where $[0, b) \in \pi$, $(a_i, b_i) \subset [0, 1)$. However, $\forall\, i \geq 1$, $(a_i, b_i) = \bigcup_{j=1}^{\infty}[a_{ij}, b_{ij})$, where $a_i < a_{ij} < b_i$, $b_{ij} = b_i$, $\forall j \geq 1$, and $\lim_{j\to\infty} a_{ij} = a_i$. Thus every open subset of $[0, 1)$ is in $\sigma(\mathcal{R})$. Now if $K \subset [0, 1)$ is a compact set, then $[0,1)\backslash K \in \sigma(\mathcal{R})$. This implies $K \in \sigma(\mathcal{R})$, hence $\mathcal{B} \subset \sigma(\mathcal{R})$. The definition of Σ will be given in Stage 3.

Stage 2. Construction of μ and m on $\mathcal{R}_0$. Let $D \subset F$ be a nonempty bounded not s-dentable set. There exists then $U_0 \in \mathfrak{U}$, and $\alpha_0 > 0$ such that

(a) U_0 is a non-s-denting neighborhood for D, i.e., $\forall d \in D$, $d \in s(D\backslash\{d + U_0\})$, and

(b) $D \subset \alpha_0 U_0$.

One defines μ and m inductively on π as follows:

(i) Let $d_0 \in D$ be arbitrary. By (a), there exists $\alpha_i^1 > 0$, $i \geq 1$, $\Sigma_{i=1}^{\infty}\alpha_i^1 = 1$, and $\{d_i^1\}_{i=1}^{\infty} \subseteq D\backslash\{d_0 + U_0\}$ such that $d_0 = \Sigma_{i=1}^{\infty}\alpha_i^1 d_i^1$. On π_1, define $\mu(A_i^1) = \alpha_i^1$, and $m(A_i^1) = \alpha_i^1 d_i^1$, $i \geq 1$.

(ii) Now suppose that μ and m have been defined on π_n such that $\forall n \in N$, and $\forall z \in N^n$, $m(A_z^n)/\mu(A_z^n) = d_z^n \in D$, $\forall A_z^n \in \pi_n$. Since D is not s-dentable, there exists $\alpha_{(z,i)}^{n+1} > 0$, $\Sigma_{i=1}^{\infty}\alpha_{(z,i)}^{n+1} = 1$, and $\{d_{(z,i)}^{n+1}\}_{i=1}^{\infty} \subseteq D\backslash\{d_z^n + U_0\}$ such that $d_z^n = \Sigma_{i=1}^{\infty}\alpha_{(z,i)}^{n+1} d_{(z,i)}^{n+1}$. Define

$$\mu(A_{(z,i)}^{n+1}) \equiv \alpha_{(z,i)}^{n+1} \quad \text{and} \quad m(A_{(z,i)}^{n+1}) \equiv \alpha_{(z,i)}^{n+1} d_{(z,i)}^{n+1}.$$

Observe that

$$m(A_{(z,i)}^{n+1})/\mu(A_{(z,i)}^{n+1}) - m(A_z^n)/\mu(A_z^n) \notin U_0. \tag{5}$$

By induction, μ and m are thus defined on all of π satisfying the crucial "horizontal" countable additivity property: $\forall A_z^n \in \pi$,

$$(6)\qquad \mu(A^n_z) = \sum_{i=1}^{\infty} \mu(A^{n+1}_{(z,i)}) \quad \text{and} \quad m(A^n_z) = \sum_{i=1}^{\infty} m(A^{n+1}_{(z,i)}).$$

Now μ and m will be defined on $\mathfrak{R}_0$ as follows: Let

$$S_1 = \bigcup_{i=1}^{n_1} A^{k_i}_{z_i} \quad \text{and} \quad S_2 = \bigcup_{i=1}^{n_2} \bigcup_{j=p_i}^{\infty} A^{r_i+1}_{(z_i,j)}$$

such that $S_1 \cap S_2 = \emptyset$. Define

$$(7)\qquad \mu(S_1) \equiv \sum_{i=1}^{n_1} \mu(A^{k_i}_{z_i}) \quad \text{and} \quad m(S_1) \equiv \sum_{i=1}^{n_1} m(A^{k_i}_{z_i}).$$

Using (6) and (7), one can define

$$(8)\qquad \begin{aligned} &(a) \quad \mu(S_2) = \sum_{i=1}^{n_2} \left[\mu(A^{r_i}_{z_i}) - \sum_{j=1}^{p_i-1} \mu(A^{r_i+1}_{(z_i,j)}) \right], \text{ and} \\ &(b) \quad m(S_2) \equiv \sum_{i=1}^{n_2} \left[m(A^{r_i}_{z_i}) - \sum_{j=1}^{p_i-1} m(A^{r_i+1}_{(z_i,j)}) \right]. \end{aligned}$$

Finally define $\mu(S_1 \cup S_2) \equiv \mu(S_1) + \mu(S_2)$, and $m(S_1 \cup S_2) \equiv m(S_1) + m(S_2)$ where the right sides of the above equations are defined by (7) and (8).

It is easy to check that μ and m are finitely additive on $\mathfrak{R}_0$, and that μ is a positive set function of bounded variation. In fact, the variation of μ is bounded by 1. Furthermore, by first proving for sets of π then for sets of $\mathfrak{R}_0$ by using the definition of m on π and the structure of $\mathfrak{R}_0$, (1), one can show that $\forall\, U \in \mathfrak{U}$, there exists $\alpha_U > 0$ such that

$$(9)\qquad p_U(m(A)) < \alpha_U \mu(A), \qquad \forall A \in \mathfrak{R}_0.$$

Stage 3. Extension of μ and m. Using the "horizontal" countable additivity property (6) and the structure of π_n, one can prove that μ is regular on π relative to $\mathfrak{R}_0$, i.e., $\forall\, A \in \pi$, and $\forall\, \epsilon > 0$, there exist $C, V \in \mathfrak{R}_0$ such that $C \subseteq A \subseteq V^0$, and $\mu(H) < \epsilon$, $\forall\, H \in \mathfrak{R}_0$, $H \subseteq V\backslash C$ (see [9, p. 11]). Then knowing the structure of $\mathfrak{R}_0$, and (1), one can prove by direct computation that μ is regular on $\mathfrak{R}_0$ relative to $\mathfrak{R}_0$ (this was termed regular R_2 in [4, p. 508]). Hence by Theorem 3 of [4, p. 510], μ is countably additive on $\mathfrak{R}_0$. By the Carathéodory extension procedure, μ can be extended to a unique finite positive measure μ_1 on $\mathcal{C}(\mu)$, the class of μ-measurable sets [5, Corollary 4, Proposition 6, p. 72, Definition 3, p. 67]. Since μ_1 is finite, $\Sigma(\mu)$, the class of μ-integrable sets coincide in this case with $\mathcal{C}(\mu)$, and μ_1

is a complete measure on $\Sigma(\mu)$ [5, Proposition 12, Definition 6, p. 75]. Furthermore, $\mathcal{R}_0$ is μ_1-dense in $\Sigma(\mu)$, i.e., $\forall\ S \in \Sigma(\mu)$, and $\forall\ \epsilon > 0$, there exists $B \in \mathcal{R}_0$ such that $\mu_1(S \triangle B) < \epsilon$ [5, Proposition 13, p. 76]. It should be emphasized that the extension procedure need not preserve regularity! Hence μ_1 need not be regular on $\Sigma(\mu)$, in fact, μ_1 *need not even be regular on* $\sigma(\mathcal{R})$, in the sense that $\forall\ S \in \sigma(\mathcal{R})$ and $\forall\ \epsilon > 0$, there exists $C,\ V \subseteq [0, 1)$ such that $\overline{C} \subseteq S \subseteq V^0$, and $\mu(H) < \epsilon$, $\forall\ H \in \sigma(\mathcal{R})$, $H \subseteq V \backslash C$, without the requirement that $C,\ V \in \sigma(\mathcal{R})$.

Let $\Sigma(\mu)$ be denoted by Σ, and μ_1 again by μ. Since μ is countably additive, (9) and Lemma 1 of [4, p. 506] implies that m is countably additive on $\mathcal{R}_0$. Since F is complete, μ is countable additive, and $\mathcal{R}_0$ is μ-dense in Σ, one can prove (analogous to the proof of Theorem 1, p. 62 of [5]) that m can be extended to a measure $m_1: \Sigma \to F$ such that $\forall\ U \in \mathfrak{U}$, there exists $\alpha_U > 0$ such that

$$(10) \qquad P_U(m_1(S)) \le \alpha_U \mu(S), \qquad \forall S \in \Sigma.$$

Since μ is of bounded variation on Σ, it follows from (10) that m_1 has bounded variation on Σ. Again m_1 need not be regular on Σ, or on $\sigma(\mathcal{R})$. Let m_1 be denoted by m.

The restriction of μ to $\mathcal{B}$ is a positive finite measure, so by Corollary 2 of [5, p. 347], $\mu: \mathcal{B} \to [0, 1]$ is a regular Borel measure (i.e., $\forall\ B \in \mathcal{B}$ and $\epsilon > 0$, there exists $C,\ V \subseteq [0, 1)$ such that $\overline{C} \subseteq B \subseteq V^0$, and $\mu(H) < \epsilon$, $\forall\ H \in \mathcal{B}$, $H \subseteq V \backslash C$). Now from (10) and Lemma 3 of [4, p. 509], $m: \mathcal{B} \to F$ is a regular Borel measure of bounded variation. However, the regularity of m on $\mathcal{B}$ will not be needed.

Thus one has constructed a measure space (Ω, Σ, μ) and a μ-continuous vector measure $m: \Sigma \to F$ of bounded variation such that the restrictions of μ and m to $\mathcal{B}$ are regular Borel measures.

Stage 4. $m \neq \mu_f$, for any $f \in L^1(\Omega, \mu; F)$. It suffices to show that $\forall\ B \in \mathcal{B}^+$, $A_B(m)$ has width at least $\frac{1}{2} U_0$. (In fact, it should be pointed out that it is not necessarily true that every $S \in \sigma(\mathcal{R})^+$, $A_S(m)$ has width at least $\frac{1}{2}U_0$.)

Let $B \in \mathcal{B}^+$. By regularity of μ on $\mathcal{B}$, one can find a compact C and an open V such that

(i) $\overline{C} \subseteq B \subseteq V^0$,

(ii) $\mu(H) < (1/16\alpha_0)\mu(B)$, $\forall\ H \in \mathcal{B}$, $H \subseteq V \backslash C$, where $D \subseteq \alpha_0 U_0$.

Now by the third and fourth inclusions in (3) and Proposition 7 of [5, p. 73], one has

(iii) $\mu(V \backslash C) \le (1/16\,\alpha_0)\mu(B)$, where $D \subseteq \alpha_0 U_0$.

Now by the structure of V, and (4), there exists $\{A_i\}_{i=1}^{\infty} \subset \pi$ such that

(iv) $C \subseteq \bigcup_{i=1}^{\infty} A_i \subseteq V$.

There exists at least one A_{i_0} such that

(11) $$\mu(A_{i_0} \setminus B)/\mu(A_{i_0}) \leq 1/8\alpha_0.$$

Let $G = A_{i_0} \cap B \in \mathcal{B}$. Then $G \subseteq B$, and $\mu(G) > 0$. Since $A_{i_0} \in \pi$, by taking the next partition of A_{i_0}, there exists disjoint $\{C_k\}_{k=1}^{\infty}$ such that $A_{i_0} = \bigcup_{k=1}^{\infty} C_k$. Claim that there exists C_n such that

(12) $$\mu(C_n \setminus B) < (1/8\alpha_0)\mu(C_n).$$

For otherwise,

$$\mu(A_{i_0} \setminus B) = \mu\left(\bigcup_{k=1}^{\infty} (C_k \setminus B)\right) > \frac{1}{8\alpha_0} \sum_{k=1}^{\infty} \mu(C_k) = \frac{1}{8\alpha_0} \mu(A_{i_0})$$

which contradicts (11).

Let $H = C_n \cap B \in \mathcal{B}$. Since C_k belongs to the next partition of A_{i_0}, by (5),

(13) $$P_{U_0}(m(A_{i_0})/\mu(A_{i_0}) - m(C_n)/\mu(C_n)) > 1.$$

Direct computation shows that

(14) $$P_{U_0}\left(\frac{m(G)}{\mu(G)} - \frac{m(A_{i_0})}{\mu(A_{i_0})}\right) < \frac{1}{4} \quad \text{and} \quad P_{U_0}\left(\frac{m(H)}{\mu(H)} - \frac{m(C_n)}{\mu(C_n)}\right) < \frac{1}{4}.$$

Thus,

$$\begin{aligned} P_{U_0}\left(\frac{m(G)}{\mu(G)} - \frac{m(H)}{\mu(H)}\right) &\geq P_{U_0}\left(\frac{m(A_{i_0})}{\mu(A_{i_0})} - \frac{m(C_n)}{\mu(C_n)}\right) \\ &\quad - P_{U_0}\left(\frac{m(A_{i_0})}{\mu(A_{i_0})} - \frac{m(G)}{\mu(G)}\right) - P_{U_0}\left(\frac{m(H)}{\mu(H)} - \frac{m(C_n)}{\mu(C_n)}\right) \\ &> 1 - 1/4 - 1/4 \quad \text{(by (13) and (14))} \\ &> 1/2. \end{aligned}$$

Thus,

(15) $$m(G)/\mu(G) - m(H)/\mu(H) \notin \tfrac{1}{2} U_0.$$

Thus $\forall\, B \in \mathcal{B}^+$, $A_B(m)$ has width at least $\frac{1}{2} U_0$, so by Theorem 2.1

and Theorem 2.2, $m \neq \mu_f$, for any $f \in L^1(\Omega, \mu; F)$. For if there exists such an f, then by Theorem 2.1 and Theorem 2.2, $\forall\ S \in \Sigma^+$, $\forall\ U \in \mathfrak{U}$, there exists $T \in \Sigma^+$, $T \subseteq S$ such that $A_T(m)$ has width at most U. This should hold in particular for every $B \in \mathfrak{B}^+ \subseteq \sigma(\mathfrak{R})^+$. In view of (15), this is not possible for $U = \frac{1}{2}U_0$. For suppose there exists $T \in \sigma(\mathfrak{R})^+$, $T \subseteq B$, such that $A_T(m)$ has width at most $\frac{1}{2}U_0$. But by the definition of $\mathcal{C}$, (2), and the last inclusion relation in (3), $T \in \mathfrak{B}^+$, and hence by (15), $A_T(m)$ has width at least $\frac{1}{2}U_0$, hence a contradiction. Q.E.D.

As a corollary, one deduces the following result of [7].

Corollary 3.1. *Every reflexive Fréchet space F has the RNP.*

Proof. If F is a reflexive Fréchet space, then by Theorem 5.6 of [11, p. 145], every bounded set $D \subseteq F$ is relatively weakly compact. By Theorem 1.1, D is s-dentable, and so by Theorem 3.1, F has RNP. Q.E.D.

Remark 3.1. The above corollary holds, in particular, if F is a nuclear Fréchet space, or separable dual of a barreled (DF)-space, or Fréchet-Montel space.

Remark 3.2. Corollary 3.1 shows that in a reflexive Fréchet space, every bounded set is dentable. Whether this remains true if F is s-dentable is still an open question. A related question is whether F has RNP iff F is dentable. These are known to be true for Banach spaces.

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