PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 2, April 1975

PERIODIC SOLUTIONS OF A SYSTEM OF NONLINEAR DIFFERENTIAL EQUATIONS

STANISŁAW SĘDZIWY[1]

ABSTRACT. The paper consists of a study of the existence of periodic solutions of a system of differential equations using Borsuk's theorem on odd mappings. Applications are given to nth order nonlinear vector differential systems and nth order nonlinear scalar differential equations.

1. **Introduction.** This note concerns the existence of ω-periodic solutions of the system of differential equations

$$y' = Ay + F(t, y), \tag{1.1}$$

where $y \in R^n$ (R^n is real Euclidean n-space), A is a constant $n \times n$ matrix and $F\colon R \times R^n \longrightarrow R^n$ is continuous, ω-periodic in t, and F is not necessarily small.

This problem in the case of a nonsingular A has been investigated by M. A. Krasnosel'skiĭ [1], V. A. Pliss [2], and others. The present paper generalizes results of the above papers to the case of matrices with multiple characteristic root $\lambda_0 = 0$. Partial results in this direction have been obtained by the author [8], [9] and R. Reissig (see [3], [4], for example) in connection with the investigation of the behavior of solutions of nth order nonlinear equations.

In contrast to previous papers, the proofs presented here are based on an application of Borsuk's theorem on odd mappings to the translation operation generated by (1.1).

The main result of the paper is stated and proved in § 2. In § 3, applications of the main result are discussed.

The following notation will be used. The norm in R^n is denoted by $|\ |$. If B is an $m \times n$ matrix, B^T denotes the matrix transpose of B. If

Received by the editors January 2, 1974.

AMS (MOS) subject classifications (1970). Primary 34C25.

Key words and phrases. Periodic solutions, Borsuk's theorem, system of ordinary nonlinear differential equations.

[1]This paper was written while the author was visiting the University of Utah, Autumn 1973.

A is square, A^{-1} denotes the inverse and $|A|$ the norm, $\max\{|Ax|: |x| = 1\}$, of A. The same notation will be used for norms in spaces of lower dimensions. I_k denotes the $k \times k$ unit matrix. If $C \subset R^n$, then $\bar{C}$, Int C, and Bd C will stand for the closure, the interior, and the boundary of C. Recall that $C \subset R^n$ is a cone (with vertex at the origin) if $\lambda C \equiv \{\lambda x: x \in C\} \subset C$ for $\lambda \geq 0$.

2. Existence of periodic solutions of first order systems.

Theorem 1. *Assume $\lambda_0 = 0$ is a k-fold ($k \leq n$) characteristic root of A, and let the remaining roots be different from $2p\pi i/\omega$, $p = 1, 2, \cdots$. Let A have k linearly independent eigenvectors $\{m_i\}_{i=1}^k$ corresponding to λ_0. Let $F: R \times R^n \to R^n$ be continuous and ω-periodic in t, and let*

$$(2.1) \qquad |F(t, y)| \leq \mu_0 |y| \quad \text{for } |y| \geq y_0 \text{ and } t \in [0, \omega].$$

Suppose there exists an $n \times k$ matrix N such that $A^T N = 0$, $\operatorname{rank}(N) = k$ and

$$(2.2) \qquad y^T N N^T F(t, y) \leq 0 \quad \text{for all } y \in C,\ |y| \geq y_1,\ t \in [0, \omega],$$

where C is a cone satisfying $\{y: y = \Sigma_{i=1}^k \lambda_i m_i,\ \Sigma_{i=1}^k |\lambda_i| > 0\} \subset \operatorname{Int} C$.

Then (1.1) has at least one ω-periodic solution provided μ_0 is sufficiently small.

Proof. Let P be a nonsingular $n \times n$ matrix having m_i ($i = 1, \cdots, k$) as its last columns. The linear change of coordinates, $y = Px$, transforms (1.1) into the system

$$(2.3) \qquad x_1' = Dx_1 + f_1(t, x), \qquad x_2' = f_2(t, x),$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x_1 \in R^{n-k}, \quad \begin{pmatrix} f_1(t, x) \\ f_2(t, x) \end{pmatrix} = f(t, x) = P^{-1} F(t, Px);$$

then (2.1) and (2.2) assume the form

$$(2.4) \qquad |f(t, x)| \leq \mu |x| \quad \text{for } |x| \geq r_0,\ t \in [0, \omega],$$

$$(2.5) \qquad x_2^T H f_2(t, x) \leq 0 \quad \text{for } |x| \geq r_1,\ t \in [0, \omega],\ x \in C_1,$$

where $\left(\begin{smallmatrix} 0 & 0 \\ 0 & H \end{smallmatrix}\right) = P^T N N^T P$ and C_1 is the cone $\{x: Px \in C\}$. Since $\operatorname{rank}(N) = k$, H is positive definite, i.e. $w^T H w > 0$ for all $w \in R^k$, $w \neq 0$. For $w \in R^k$ let $\|w\|^2 = w^T H w$, and for $\gamma > 0$ put $C(\gamma) = \{x: |x_1| < \gamma \|x_2\|\}$. Since Int C_1

contains the set $\{x:\ x_1 = 0,\ x_2 \neq 0\}$, a compactness argument with the set $|x_2| = 1$ shows $C(\gamma) \subset \operatorname{Int}(C_1)$ for all small $\gamma > 0$. Fix γ.

Assume for the moment that f_2 satisfies (2.5) with the strict inequality sign, and let initial value problems for (2.3) have unique solutions. Associated with (2.3) is the mapping T of R^n into itself (the *translation operator* of (2.3)) defined by $T(x_0) = x(\omega, x_0)$, where $x = x(t, x_0) = (x_1(t, x_0), x_2(t, x_0))$ denotes the solution of the initial value problem (2.3), with initial conditions $x_1(0) = x_1^0$, $x_2(0) = x_2^0$ $(x_0 = (x_1^0, x_2^0))$. Clearly T is continuous, and (2.3) has an ω-periodic solution if and only if T has a fixed point. For the existence proof we will construct an open bounded set $G \subset R^n$ such that $0 \in G$, $G = -1 \cdot G$ (i.e. G is *symmetric*) and

$$(2.6) \qquad (T - e)(x) \neq 0 \quad \text{for } x \in \operatorname{Bd} G,$$

$$(2.7) \qquad \beta(T - e)(x) \neq (1 - \beta)(T - e)(-x) \quad \text{for } x \in \operatorname{Bd} G,\ \beta \in [1/2, 1],$$

where e is the identity map. By Borsuk's theorem [7, Corollary 3.31, p. 82], this will imply that T has a fixed point in G.

First we will prove that there exist positive numbers $\mu_1, \gamma_1, r_2, r_3$ such that, for all $\mu \in [0, \mu_1]$, if $\|x_2^0\| \geq r_2$ and $x_0 \in C(\gamma_1)$ then

$$(2.8) \qquad |x(t, x_0)| > r_1 \quad \text{and} \quad x(t, x_0) \in C(\gamma) \quad \text{for } t \in [0, \omega],$$

and, if $|x_1^0| \geq r_3$ and $x_0 \notin C(\gamma_1)$, then

$$(2.9) \qquad x_1(\omega, x_0) \neq x_1^0.$$

In order to see this, note first that (2.4) implies that

$$(2.10) \qquad |f(t, x)| \leq \alpha + \mu|x| \quad \text{for } (t, x) \in R \times R^n,$$

where α is a suitable constant.

Let $c = \max\{|e^{Dt}| + 1:\ t \in [0, \omega]\}$. By (2.3),

$$(2.11) \qquad \begin{aligned} x_1(t, x_0) &= e^{Dt}x_1^0 + \int_0^t e^{D(t-s)}f_1(s, x_0))\,ds, \\ x_2(t, x_0) &= x_2^0 + \int_0^t f_2(s, x(s, x_0))\,ds. \end{aligned}$$

Since the norm of the matrix

$$\begin{pmatrix} e^{Dt} & 0 \\ 0 & I_k \end{pmatrix}$$

does not exceed c, (2.10) and (2.11) imply that

$$|x(t, x_0)| \le c|x_0| + \int_0^t c(\alpha + \mu|x(s, x_0)|)\, ds.$$

Hence by Gronwall's inequality

$$|x(t, x_0)| \le c(|x_0| + \alpha\omega)e^{c\mu t} \quad \text{for } t \in [0, \omega]. \tag{2.12}$$

Let $c_1 = \max\{|e^{Dt} - I_{n-k}|: t \in [0, \omega]\}$, $y_i = x_i(t, x_0)$ $(i = 1, 2)$, and denote by h_1^2, h_2^2, respectively, the largest and the smallest eigenvalue of H.

From (2.12) and inequalities

$$|y - x_0| \le |y_1 - x_1^0| + |y_2 - x_2^0|,$$

$$|y_1 - x_1^0| \le c_1|x_1^0| + \int_0^t c(\alpha + \mu|x(s, x_0)|)\, ds,$$

$$|y_2 - x_2^0| \le \int_0^t (\alpha + \mu|x(s, x_0)|)\, ds,$$

it follows that

$$|y - x_0| \le k_1(\mu)|x_1^0| + (k_2(\mu) + k_3(\mu)r_2^{-1})|x_2^0| \quad \text{for } t \in [0, \omega],\ |x_2^0| \ge r_2,$$

where $k_2(\mu) = (c + 1)(e^{\mu c\omega} - 1)$, $k_1(\mu) = c_1 + k_2(\mu)$ and $k_3(\mu) = \alpha\omega[1 + c + k_2(\mu)]$.

If $x_0 \in C(\gamma_1)$, then $|x_1^0| \le \gamma_1 h_1 |x_2^0|$. The preceding inequalities together with $|y_1| \le |y - x_0| + |x_1^0|$, $|x_2^0| \le |y - x_0| + |y_2|$ imply

$$|y_1| \le [(k_1(\mu) + 1)\gamma_1 h_1 + k_2(\mu) + k_3(\mu)r_2^{-1}]|x_2^0| \equiv m_1|x_2^0|,$$

$$|x_2^0|[1 - k_1(\mu)\gamma_1 h_1 - k_2(\mu) - k_3(\mu)r_2^{-1}] \equiv |x_2^0|m_2 \le |y_2|.$$

Let r_2 be so large that $r_2 > h_1 r_1 + 2k_3(0)$ and $k_3(0)[r_2 - k_3(0)]^{-1} < \gamma h_2$.

Then, by continuity, there exist $\gamma_1 > 0$ and $\mu_0 > 0$ such that for $\mu \in [0, \mu_1]$

$$m_1 \cdot m_2^{-1} < \gamma h_2, \qquad h_1^{-1}(m_2 - m_1)r_2 > r_1,$$

which proves (2.8).

To show (2.9), note that $x_1(\omega, x_0) - x_1^0 = 0$ is equivalent to

$$(e^{D\omega} - I_{n-k})x_1^0 = -\int_0^\omega e^{D(\omega - s)} f_1(s, x(s, x_0))\, ds.$$

Since characteristic roots of D are different from $2p\pi i/\omega$ $(p = 1, 2, \cdots)$, the matrix $e^{D\omega} - I_{n-k}$ is nonsingular, which implies that there is an $l > 0$ such that $l|w| \le |(e^{D\omega} - I_{n-k})w|$ for all $w \in R^{n-k}$.

Since $\|x_2\| \geq h_2 |x_2|$, $x_0 \notin C(\gamma_1)$ implies that $\gamma_1 h_2 |x_2^0| \leq |x_1^0|$. Hence $|x_0| < (1 + 1/h_2\gamma_1)|x_1^0|$ and we get finally

$$(2.13) \qquad l|x_1^0| \leq c\alpha\omega e^{\mu c\omega} + c\left(1 + \frac{1}{h_2\gamma_1}\right)|x_1^0|(e^{\mu c\omega} - 1).$$

Replacing μ_1 by a smaller number if necessary, we may assume that $c(1 + 1/h_2\gamma_1)(e^{\mu_1 c\omega} - 1) < l$. So from (2.13) it follows that if x_1^0 satisfies $x_1(\omega, x_0) - x_1^0 = 0$, then $|x_1^0|$ is bounded, which proves (2.9).

Now let G be the set $\{x: |x_1| < r_3\} \cap \{x: \|x_2\| < r_2\}$ where r_2 and r_3 are defined as above. We have $\mathrm{Bd}\, G = F_1 \cup F_2$, where $F_1 = \{x: |x_1| = r_3,$ $\|x_2\| \leq r_2\}$ and $F_2 = \{x: |x_1| < r_3, \|x_2\| = r_2\}$.

Increasing r_2 or r_3 if necessary, we may assume that $r_3 = \gamma_1 r_2$. Then $\gamma_1 \|x_2^0\| \leq |x_1^0|$ for $x_0 \in F_1$, hence $x_0 \notin C(\gamma_1)$. Thus (2.9) implies (2.6) for $x_0 \in F_1$.

Let $x_0 \in F_2$. Put $W(t) = \|x_2(t, x_0)\|$. Since $F_2 \subset C(\gamma_1)$ and

$$W'(t) = \frac{1}{W(t)} x_2^T(t, x_0) H f_2(t, x(t, x_0)),$$

(2.5) and (2.8) imply that $W'(t) < 0$ for $t \in [0, \omega]$; hence

$$(2.14) \qquad \|x_2(\omega, x_0)\| < \|x_2^0\|,$$

which completes the proof of (2.6).

From (2.14) it is easy to verify (2.7), for $x_0 \in F_2$.

Assume $x_0 \in F_1$, and (2.7) fails for some $\beta \in [½, 1]$. By (2.11),

$$|(e^{D\omega} - I_{n-k})x_1^0| \leq \beta \int_0^\omega c(\alpha + \mu|x(s, x_0)|)\, ds$$
$$+ (1 - \beta) \int_0^\omega c(\alpha + \mu|x(s, -x_0)|)\, ds,$$

and by arguments used in the proof of (2.9), we conclude that $|x_1^0| < r_3$, which contradicts $x_0 \in F_1$.

Thus G has properties (2.6), (2.7), and reference to Borsuk's theorem proves the theorem under our special assumptions of uniqueness and strict inequality in (2.5).

The general case is obtained by uniform approximation of f on $[0, \omega] \times \overline{G}$ and a standard limiting argument (see [10], for example). The details are left to the reader.

3. **Periodic solutions of *n*th order systems.** As an application of Theorem 1, we will consider the problem of existence of periodic solutions

of the nth order system of differential equations

(3.1) $$y^{(n)} + A_1 y^{(n-1)} + \cdots + A_{n-1} y' + f(y) = p(t),$$

where y is an m-vector, the A_i are constant $m \times m$ matrices, and f: $R^m \to R^m$, p: $R \to R^m$ are continuous.

Theorem 2. *Let A_{n-1} be symmetric and positive definite and let the roots of the polynomial $\phi(\lambda) = \det(\lambda^{n-1} I_m + \lambda^{n-2} A_1 + \cdots + A_{n-1})$ satisfy $\lambda_i \neq (2\pi i/\omega)p$ $(p = 1, 2, \cdots)$. Assume that*

(3.2) $$|f(y)| \leq \mu_0 |y| \quad \textit{for } |y| \geq y_0,$$

(3.3) $$y^T f(y) \geq k|y||f(y)| \qquad (0 < k \leq 1) \textit{ for all } y.$$

Let p be ω-periodic and satisfy one of the conditions:

(3.4) $$\int_0^\omega p(s)\, ds = 0,$$

or

(3.5) $$\min\{k|f(y)|: |y| \geq y_1\} > p_2 = \max\{|p(t)|: t \in [0, \omega]\}.$$

Then (3.1) *has an ω-periodic solution, provided μ_0 is small enough.*

A weaker version of this theorem was proved in [9]. Theorem 2 with (3.4) gives an extension of results of R. Reissig [3], [4] where the existence of periodic solutions was proved under the hypotheses that $\lim_{|y|\to\infty} |f(y)|/|y| = 0$ and that ϕ has roots with negative real parts.

Proof. Let (3.4) hold. Replace (3.1) by the equivalent system

(3.6) $$x' = Fx + bz + bp_1(t), \qquad z' = -f(c^T x),$$

where $p_1(t) = \int_0^t p(s)\, ds$, $x^T = (x_1 \cdots x_{n-1})$, $x_i \in R^m$, $x_1 = y$ and b, c, F are the $(n-1)m \times m$, $(n-1)m \times m$, $(n-1)m \times (n-1)m$ matrices

$$b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_m \end{pmatrix}, \quad c = \begin{pmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ -A_{n-1} & -A_{n-2} & -A_{n-3} & \cdots & -A_1 \end{pmatrix}.$$

Then (3.6) is of the form (1.1) with

$$A = \begin{pmatrix} F & b \\ 0 & 0 \end{pmatrix}, \qquad F(t, x, z) = \begin{pmatrix} bp_1(t) \\ -f(c^T x) \end{pmatrix},$$

so it suffices to verify the assumptions of Theorem 1. Put

$$M = \begin{pmatrix} -F^{-1}b \\ I_m \end{pmatrix}.$$

Since $\det(A - \lambda I_{mn}) = \lambda^m (-1)^{m(n-1)} \phi(\lambda)$, $\operatorname{rank}(M) = m$, $AM = 0$, the matrix A satisfies the conditions of Theorem 1.

By (3.2) and (3.3), $F(t, x, z)$ satisfies (2.1). Let K be a symmetric matrix such that $K^2 = A_{n-1}^{-1}$. Define the $nm \times m$ matrix N by $N = \binom{0}{K}$. With the above notation, (2.2) may be written as

(3.7) $$-z^T K^2 f(c^T x) \le 0.$$

By (3.3),

$$z^T K^2 f(c^T x) = (K^2 z - c^T x + c^T x)^T f(c^T x) \ge |f(c^T x)|(k|c^T x| - |K^2 z - c^T x|),$$

so (3.7) holds for (x, z) belonging to the cone $C = \{(x, z): |K^2 z - c^T x| \le k|c^T x|\}$.

Observe that $L = \{(x, z): x = -F^{-1}bw,\ z = w,\ w \in R^m\}$ is a subspace of R^{mn} spanned by eigenvectors of A corresponding to $\lambda = 0$. Since $-c^T F^{-1} b = A_{n-1}^{-1}$, then $K^2 + c^T F^{-1} b = 0$, and direct calculation yields $(L \setminus \{0\}) \subset \operatorname{Int} C$, which proves Theorem 2 with the assumption (3.4).

In the case (3.5), replace (3.1) by

(3.8) $$x' = Fx + bz, \qquad z' = -f(c^T x) + p(t),$$

where x, b, c, F are as before.

Since (3.8) is obtained from (3.6) by taking $p_1(t) = 0$ and replacing $f(c^T x)$ by $f_2(t, x, z) = f(c^T x) - p(t)$, it remains only to verify that f_2 satisfies (2.2).

By (3.5), there is a $k_1 \in (0, k)$ such that $y^T(f(y) - p(t)) \ge (k - k_1)|y| \cdot |f(y)|$, which implies that $-z^T K^2 (f(c^T x) - p(t)) \le 0$ for (x, z) in the cone $\{(x, z): |K^2 z - c^T x| \le ((k - k_1)/(1 + k))\, |c^T x|\}$; this completes the proof of Theorem 2.

In the scalar case, instead of (3.1) one can consider a slightly more general equation

(3.9) $y^{(n)}+g_1(y^{(n-2)})y^{(n-1)}+\cdots+g_{n-1}(y)y'+f(y;\, y',\, \cdots,\, y^{(n-1)})=p(t),$

where $g_i\colon R\to R$, $f\colon R\times R^{n-1}\to R$, $p\colon R\to R$ are continuous. An application of Theorem 2 gives

Theorem 3. *Let $G_i(u)=\int_0^u g_i(s)\,ds$ satisfy*

(3.10) $$|G_i(u)-a_iu|\le\mu|u| \quad \text{for } i=1,\cdots,n-1,\ |u|\ge u_1.$$

Assume that $a_{n-1}>0$ and that the polynomial $\phi(\lambda)=\lambda^{n-1}+a_1\lambda^{n-2}+\cdots+a_{n-1}$ has roots different from $(2\pi i/\omega)p$, $p=1,2,\cdots$. Let

(3.11) $$0<yf(y,x)<\mu y^2 \quad \text{for } |y|\ge y_1,\ z\in R^{n-1}.$$

Let p be ω-periodic and either

(3.12) $$\int_0^\omega p(s)\,ds=0$$

or

(3.13) $$|f(y,z)|\ge f_0>\max\{|p(t)|\colon t\in[0,\omega]\} \quad \text{for } |y|\ge y_1,\ z\in R^{n-1}.$$

If μ is sufficiently small, then (3.9) has at least one ω-periodic solution.

Proof. Let $x_1=y$, $x^T=(x_1,\cdots,x_{n-1})$ and
$$G(x)=[a_{n-1}x_1-G_{n-1}(x_1)]+\cdots+[a_1x_{n-1}-G_1(x_{n-1})],$$
and let b, c, F be as in Theorem 2 ($m=1$).

Then (3.9) may be replaced by

(3.14) $$x'=Fx+bz+F_1(t,x,z), \qquad z'=F_2(t,x,z),$$

where

$$F_1(t,x,z)=b\int_0^t p(s)\,ds+bG(x),$$

$$F_2(t,x,z)=-f(c^Tx;\, Fx+bz+F_1(t,x,z))$$

if p satisfies (3.13), and

$$F_1(t,x,z)=bG(x), \qquad F_2(t,x,z)=-f(c^Tx;\, Fx+bz+F_1(t,x,z))+p(t)$$

otherwise. It is easily seen that the proof repeats arguments used above, hence it is left to the reader.

Theorem 3 generalizes results of [5] and partially of [6].

REFERENCES

1. M. A. Krasnosel'skiĭ, *The operator of translation along the trajectories of differential equations*, "Nauka", Moscow, 1966; English transl., Transl. Math. Monographs, vol. 19, Amer. Math. Soc., Providence, R.I., 1968. MR 34 #3012; 36 #6688.

2. V. A. Pliss, *The existence of periodic solutions of certain nonlinear systems*, Dokl. Akad. Nauk SSSR 137 (1961), 1060–1062 = Soviet Math. Dokl. 2 (1961), 416–418. MR 22 #9665.

3. R. Reissig, *Note on a certain non-autonomous differential equation*, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur. (8) 48 (1970), 484–486. MR 44 #1879.

4. ———, *Periodic solutions of a nonlinear n-th order vector differential equation*, Ann. Mat. Pura Appl. (4) **87** (1970), 111–123. MR 44 #542.

5. R. Reissig, *Periodic solutions of a third order nonlinear differential equation*, Ann. Mat. Pura. Appl. 92 (1972), 193–198.

6. ———, *An extension of Ezeilo's result*, Ann. Mat. Pura. Appl. **92** (1972), 199–210.

7. J. T. Schwartz, *Nonlinear functional analysis*, Gordon and Breach, New York, 1969.

8. S. Sedziwy, *Asymptotic properties of solutions of nonlinear differential equations of the higher order*, Zeszyty Nauk. Uniw. Jagiello. Prace Mat. 11 (1966), 69–80. MR 44 #547.

9. ———, *Asymptotic properties of solutions of a certain n-th order vector differential equation*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 47 (1969), 472–475. MR 42 #6366.

10. ———, *Periodic solutions of differential equations in the cylindrical space*, Ann. Polon. Math. 26 (1972), 335–339. MR 46 #7637.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112

Current address: Instytut Matematyczny, Uniwersytet Jagielloński, Kraków, Reymonta 4, Poland