PERIODIC SOLUTIONS OF A SYSTEM OF NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The paper consists of a study of the existence of periodic solutions of a system of differential equations using Borsuk's theorem on odd mappings. Applications are given to nth order nonlinear vector differential systems and nth order nonlinear scalar differential equations.

1. Introduction. This note concerns the existence of \( \omega \)-periodic solutions of the system of differential equations

\[
y' = Ay + F(t, y),
\]

where \( y \in \mathbb{R}^n \) (\( \mathbb{R}^n \) is real Euclidean \( n \)-space), \( A \) is a constant \( n \times n \) matrix and \( F: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous, \( \omega \)-periodic in \( t \), and \( F \) is not necessarily small.

This problem in the case of a nonsingular \( A \) has been investigated by M. A. Krasnosel'skiĭ [1], V. A. Pliss [2], and others. The present paper generalizes results of the above papers to the case of matrices with multiple characteristic root \( \lambda_0 = 0 \). Partial results in this direction have been obtained by the author [8], [9] and R. Reissig (see [3], [4], for example) in connection with the investigation of the behavior of solutions of 72th order nonlinear equations.

In contrast to previous papers, the proofs presented here are based on an application of Borsuk's theorem on odd mappings to the translation operation generated by (1.1).

The main result of the paper is stated and proved in § 2. In § 3, applications of the main result are discussed.

The following notation will be used. The norm in \( \mathbb{R}^n \) is denoted by \( | \cdot | \). If \( B \) is an \( m \times n \) matrix, \( B^T \) denotes the matrix transpose of \( B \). If
A is square, $A^{-1}$ denotes the inverse and $|A|$ the norm, \( \max \{|Ax|: |x| = 1\} \), of $A$. The same notation will be used for norms in spaces of lower dimensions. $I_k$ denotes the $k \times k$ unit matrix. If $C \subset \mathbb{R}^n$, then $\overline{C}$, $\text{Int } C$, and $\text{Bd } C$ will stand for the closure, the interior, and the boundary of $C$. Recall that $C \subset \mathbb{R}^n$ is a cone (with vertex at the origin) if $\lambda \overline{C} = \{ \lambda x: x \in C \} \subset C$ for $\lambda \geq 0$.

2. Existence of periodic solutions of first order systems.

**Theorem 1.** Assume $\lambda_0 = 0$ is a $k$-fold ($k \leq n$) characteristic root of $A$, and let the remaining roots be different from $\frac{2p\pi}{\omega}$, $p = 1, 2, \ldots$. Let $A$ have $k$ linearly independent eigenvectors $\{m_i\}_{i=1}^k$ corresponding to $\lambda_0$. Let $F: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and $\omega$-periodic in $t$, and let

\[
|F(t, y)| \leq \mu_0 |y| \quad \text{for } |y| \geq y_0 \text{ and } t \in [0, \omega].
\]

Suppose there exists an $n \times k$ matrix $N$ such that $A N = 0$, $\text{rank}(N) = k$ and

\[
y^TNN^TF(t, y) \leq 0 \quad \text{for all } y \in C, |y| \geq y_1, t \in [0, \omega],
\]

where $C$ is a cone satisfying $\{ y: y = \sum_{i=1}^k \lambda_i m_i, \sum_{i=1}^k |\lambda_i| > 0 \} \subset \text{Int } C$.

Then (1.1) has at least one $\omega$-periodic solution provided $\mu_0$ is sufficiently small.

**Proof.** Let $P$ be a nonsingular $n \times n$ matrix having $m_i$ ($i = 1, \ldots, k$) as its last columns. The linear change of coordinates, $y = Px$, transforms (1.1) into the system

\[
x_1' = Dx_1 + f_1(t, x), \quad x_2' = f_2(t, x),
\]

where

\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x_1 \in \mathbb{R}^{n-k}, \quad \begin{pmatrix} f_1(t, x) \\ f_2(t, x) \end{pmatrix} = f(t, x) = P^{-1}F(t, Px);
\]

then (2.1) and (2.2) assume the form

\[
|f(t, x)| \leq \mu |x| \quad \text{for } |x| \geq r_0, t \in [0, \omega],
\]

\[
x_2^T H f_2(t, x) \leq 0 \quad \text{for } |x| \geq r_1, t \in [0, \omega], x \in C_1,
\]

where \( \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix} = P^T N N^T P \) and $C_1$ is the cone $\{ x: Px \in C \}$. Since $\text{rank}(N) = k$, $H$ is positive definite, i.e. $w^T H w > 0$ for all $w \in \mathbb{R}^k$, $w \neq 0$. For $w \in \mathbb{R}^k$ let $\|w\|^2 = w^T H w$, and for $\gamma > 0$ let $C(\gamma) = \{ x: |x_1| < \gamma \|x_2\| \}$. Since $\text{Int } C_1$
contains the set \{x: x_1 = 0, x_2 \neq 0\}, a compactness argument with the set |
x_2| = 1 shows \(C(y) \subset \text{Int}(C_1)\) for all small \(y > 0\). Fix \(y\).

Assume for the moment that \(f_2\) satisfies (2.5) with the strict inequality sign, and let initial value problems for (2.3) have unique solutions. Associated with (2.3) is the mapping \(T\) of \(R^n\) into itself (the translation operator of (2.3)) defined by \(T(x_0) = x(\omega, x_0)\), where \(x = x(t, x_0) = (x_1(t, x_0), x_2(t, x_0))\) denotes the solution of the initial value problem (2.3), with initial conditions \(x_1(0) = x_1^0, x_2(0) = x_2^0 (x_0 = (x_1^0, x_2^0))\). Clearly \(T\) is continuous, and (2.3) has an \(\omega\)-periodic solution if and only if \(T\) has a fixed point. For the existence proof we will construct an open bounded set \(G \subset R^n\) such that \(0 \in G, G = -G\) (i.e. \(G\) is symmetric) and

\[(T - e)(x) \neq 0\quad \text{for } x \in \text{Bd } G,\]

\[(T - e)(x) \neq (1 - \beta)(T - e)(-x)\quad \text{for } x \in \text{Bd } G, \beta \in [\frac{1}{2}, 1],\]

where \(e\) is the identity map. By Borsuk’s theorem [7, Corollary 3.31, p. 82], this will imply that \(T\) has a fixed point in \(G\).

First we will prove that there exist positive numbers \(\mu_1, \gamma_1, r_2, r_3\) such that, for all \(\mu \in [0, \mu_1]\), if \(\|x^0_2\| \geq r_2\) and \(x_0 \in C(\gamma_1)\) then

\[|x(t, x_0)| > r_1\quad \text{and } x(t, x_0) \in C(y)\quad \text{for } t \in [0, \omega],\]

and, if \(|x^0_1| \geq r_3\) and \(x_0 \notin C(\gamma_1)\), then

\[x_1(\omega, x_0) \neq x_1^0.\]

In order to see this, note first that (2.4) implies that

\[|f(t, x)| \leq \alpha + \mu|x|\quad \text{for } (t, x) \in R \times R^n,\]

where \(\alpha\) is a suitable constant.

Let \(c = \max \{\|e^{Dt}\| + 1: t \in [0, \omega]\}\). By (2.3),

\[x_1(t, x_0) = e^{Dt}x_1^0 + \int_0^t e^{D(t-s)}f_1(s, x(s))\,ds,\]

\[x_2(t, x_0) = x_2^0 + \int_0^t f_2(s, x(s, x_0))\,ds.\]

Since the norm of the matrix

\[
\begin{pmatrix}
e^{Dt} & 0 \\
0 & I_k
\end{pmatrix}
\]

does not exceed \(c\), (2.10) and (2.11) imply that
\[ |x(t, x_0)| \leq c|x_0| + \int_0^t c(\alpha + \mu|x(s, x_0)|)\,ds. \]

Hence by Gronwall's inequality

(2.12) \[ |x(t, x_0)| \leq c(|x_0| + \alpha \omega)e^{c\mu t} \quad \text{for} \quad t \in [0, \omega]. \]

Let \( c_1 = \max \{|e^{D_1 t} - I_{n-k}| : t \in [0, \omega]\}, \gamma_i = x_i(t, x_0) \quad (i = 1, 2), \) and denote by \( h_{1}^2, h_{2}^2, \) respectively, the largest and the smallest eigenvalue of \( H. \)

From (2.12) and inequalities

\[ |y - x_0| \leq |y_1 - x_1^0| + |y_2 - x_2^0|, \]
\[ |y_1 - x_1^0| \leq c_1|x_0^0| + \int_0^t c(\alpha + \mu|x(s, x_0)|)\,ds, \]
\[ |y_2 - x_2^0| \leq \int_0^t (\alpha + \mu|x(s, x_0)|)\,ds, \]

it follows that

\[ |y - x_0| \leq k_1(\mu)|x_1^0| + (k_2(\mu) + k_3(\mu)r_2^{-1})|x_2^0| \quad \text{for} \quad t \in [0, \omega], \quad |x_2^0| \geq r_2, \]

where \( k_2(\mu) = (c + 1)(e^{\mu c\omega} - 1), \quad k_1(\mu) = c_1 + k_2(\mu) \) and \( k_3(\mu) = \alpha \omega [1 + c + k_2(\mu)]. \)

If \( x_0 \in C(\gamma_1), \) then \( |x_1^0| \leq \gamma_1 h_1 |x_2^0|. \) The preceding inequalities together with

\[ |y_1| \leq |y - x_0| + |x_1^0|, \quad |x_2^0| \leq |y - x_0| + |y_2| \]

imply

\[ |y_1| \leq [(k_1(\mu) + 1)\gamma_1 b_1 + k_2(\mu) + k_3(\mu)r_2^{-1}]|x_2^0| \equiv m_1|x_2^0|, \]
\[ |x_2^0|[1 - k_1(\mu)\gamma_1 b_1 - k_2(\mu) - k_3(\mu)r_2^{-1}] \equiv |x_2^0| m_2 \leq |y_2|. \]

Let \( r_2 \) be so large that \( r_2 > h_1 r_1 + 2k_3(0) \) and \( k_3(0)[r_2 - k_3(0)]^{-1} < \gamma h_2. \)

Then, by continuity, there exist \( \gamma_1 > 0 \) and \( \mu_0 > 0 \) such that for \( \mu \in \left[0, \mu_1\right] \)

\[ m_1 \cdot m_2^{-1} < \gamma h_2, \quad h_1^{-1}(m_2 - m_1)r_2 > r_1, \]

which proves (2.8).

To show (2.9), note that \( x_1(\omega, x_0) - x_1^0 = 0 \) is equivalent to

\[ (e^{D_1 \omega - I_{n-k}})x_1^0 = -\int_0^\omega e^{D(\omega-s)}f_1(s, x(s, x_0))\,ds. \]

Since characteristic roots of \( D \) are different from \( 2pni/\omega \) \( (p = 1, 2, \cdots), \)

the matrix \( e^{D_1 \omega - I_{n-k}} \) is nonsingular, which implies that there is an \( l > 0 \)

such that \( |w| \leq |(e^{D_1 \omega - I_{n-k}})w| \) for all \( w \in R^{n-k}. \)
Since \( \|x_2\| \geq h_1|x_2| \), \( x_0 \not\in C(y_1) \) implies that \( y_1 h_2|x_2^0| \leq |x_1^0| \). Hence \( |x_0| < \left(1 + 1/h_2 y_1\right)|x_0^0| \) and we get finally
\[
(2.13) \quad \|x_1^0\| \leq c \alpha \omega e^{\mu \epsilon \omega} + c \left(1 + \frac{1}{h_2 y_1}\right)|x_0^0|(e^{\alpha \epsilon \omega} - 1).
\]

Replacing \( \mu_1 \) by a smaller number if necessary, we may assume that \( c(1 + 1/h_2 y_1)(e^{\mu_1 \epsilon \omega} - 1) < l \). So from (2.13) it follows that if \( x_1^0 \) satisfies \( x_1(\omega, x_0) - x_1^0 = 0 \), then \( |x_1^0| \) is bounded, which proves (2.9).

Now let \( G \) be the set \( \{x: |x_1| < r_2, \|x_2\| < r_2\} \) where \( r_2 \) and \( r_3 \) are defined as above. We have \( \partial G = F_1 \cup F_2 \), where \( F_1 = \{x: |x_1| = r_3, \|x_2\| \leq r_2\} \) and \( F_2 = \{x: |x_1| < r_3, \|x_2\| = r_2\} \).

Increasing \( r_2 \) or \( r_3 \) if necessary, we may assume that \( r_3 = y_1 r_2 \). Then \( \gamma \|x_2^0\| \leq |x_1^0| \) for \( x_0 \in F_1 \), hence \( x_0 \not\in C(y_1) \). Thus (2.9) implies (2.6) for \( x_0^1 \in F_1 \).

Let \( x_0 \in F_2 \). Put \( W(t) = \|x_2(t, x_0)\| \). Since \( F_2 \subset C(y_1) \) and
\[
(2.5) \quad W'(t) = \frac{1}{W(t)} x_2^T(t, x_0) H_2(t, x(t, x_0)),
\]
(2.5) and (2.8) imply that \( W'(t) < 0 \) for \( t \in [0, \omega) \); hence
\[
(2.14) \quad \|x_2(\omega, x_0)\| < \|x_2^0\|,
\]
which completes the proof of (2.6).

From (2.14) it is easy to verify (2.7), for \( x_0 \in F_2 \).

Assume \( x_0 \in F_1 \), and (2.7) fails for some \( \beta \in \left[\frac{1}{2}, 1\right] \). By (2.11),
\[
|e^{D_1 x_0^0|} - I_{n-k}|x_1^0| \leq \beta \int_0^\omega c(\alpha + \mu|x(s, x_0)|) ds
+ (1 - \beta) \int_0^\omega c(\alpha + \mu|x(s, x_0)|) ds,
\]
and by arguments used in the proof of (2.9), we conclude that \( |x_1^0| < r_3 \), which contradicts \( x_0 \in F_1 \).

Thus \( G \) has properties (2.6), (2.7), and reference to Borsuk's theorem proves the theorem under our special assumptions of uniqueness and strict inequality in (2.5).

The general case is obtained by uniform approximation of \( f \) on \([0, \omega] \times \bar{G}\) and a standard limiting argument (see [10], for example). The details are left to the reader.

3. Periodic solutions of \( n \)th order systems. As an application of Theorem 1, we will consider the problem of existence of periodic solutions
of the $n$th order system of differential equations

$$y^{(n)} + A_1 y^{(n-1)} + \cdots + A_{n-1} y' + f(y) = p(t),$$

where $y$ is an $m$-vector, the $A_i$ are constant $m \times m$ matrices, and $f: \mathbb{R}^m \rightarrow \mathbb{R}^m$, $p: \mathbb{R} \rightarrow \mathbb{R}^m$ are continuous.

**Theorem 2.** Let $A_{n-1}$ be symmetric and positive definite and let the roots of the polynomial $\phi(\lambda) = \det(\lambda^{n-1} I_m + \lambda^{n-2} A_1 + \cdots + A_{n-1})$ satisfy $\lambda_i \neq (2\pi i/\omega) p$ ($p = 1, 2, \cdots$). Assume that

$$\lambda \geq y_0,$$

$$y^T f(y) > k|y| f(y) \quad (0 < k \leq 1) \text{ for all } y.$$

Let $p$ be $\omega$-periodic and satisfy one of the conditions:

$$\int_0^\omega p(s) \, ds = 0,$$

or

$$\min \{ |f(y)| : |y| \geq y_1 \} > p_2 = \max \{ |p(t)| : t \in [0, \omega] \}.$$

Then (3.1) has an $\omega$-periodic solution, provided $\mu_0$ is small enough.

A weaker version of this theorem was proved in [9]. Theorem 2 with (3.4) gives an extension of results of R. Reissig [3], [4] where the existence of periodic solutions was proved under the hypotheses that $\lim_{|y| \rightarrow \infty} |f(y)|/|y| = 0$ and that $\phi$ has roots with negative real parts.

**Proof.** Let (3.4) hold. Replace (3.1) by the equivalent system

$$x' = Fx + bz + bp_1(t), \quad z' = -f(c^T x),$$

where $p_1(t) = \int_0^t g(s) \, ds$, $x^T = (x_1 \cdots x_{n-1})$, $x_i \in \mathbb{R}^m$, $x_1 = y$ and $b, c, F$ are the $(n-1)m \times m$, $(n-1)m \times m$, $(n-1)m \times (n-1)m$ matrices

$$b = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} I_m \\ \vdots \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ -A_{n-1} & -A_{n-2} & -A_{n-3} & \cdots & -A_1 \end{pmatrix}.$$
Then (3.6) is of the form (1.1) with

\[
A = \begin{pmatrix} F & b \\ 0 & 0 \end{pmatrix}, \quad F(t, x, z) = \begin{pmatrix} b p_1(t) \\ -f(c^T x) \end{pmatrix},
\]

so it suffices to verify the assumptions of Theorem 1. Put

\[
M = \begin{pmatrix} -F^{-1}b \\ I_m \end{pmatrix}.
\]

Since \( \det(A - \lambda I_{mn}) = \lambda^m(-1)^m(n-1)\phi(\lambda) \), rank \((M) = m\), \( AM = 0 \), the matrix \( A \) satisfies the conditions of Theorem 1.

By (3.2) and (3.3), \( F(t, x, z) \) satisfies (2.1). Let \( K \) be a symmetric matrix such that \( K^2 = A_{n-1}^{-1} \). Define the \( nm \times m \) matrix \( N \) by \( N = (0_K) \).

With the above notation, (2.2) may be written as

\[
(3.7) -z^T K^2 f(c^T x) \leq 0.
\]

By (3.3),

\[
z^T K^2 f(c^T x) = (K^2 z - c^T x + c^T x)^T f(c^T x) \geq |f(c^T x)|(k|c^T x| - |K^2 z - c^T x|),
\]

so (3.7) holds for \((x, z)\) belonging to the cone \( C = \{(x, z): |K^2 z - c^T x| \leq k|c^T x|\} \).

Observe that \( L = \{(x, z): x = -F^{-1}bw, z = w, w \in R^m\} \) is a subspace of \( R^{mn} \) spanned by eigenvectors of \( A \) corresponding to \( \lambda = 0 \). Since \(-c^T F^{-1}b = A_{n-1}^{-1}\), then \( K^2 + c^T F^{-1}b = 0 \), and direct calculation yields \((L \setminus \{0\}) \subset \text{Int } C\), which proves Theorem 2 with the assumption (3.4).

In the case (3.5), replace (3.1) by

\[
(3.8) x' = Fx + bz, \quad z' = -f(c^T x) + p(t),
\]

where \( x, b, c, F \) are as before.

Since (3.8) is obtained from (3.6) by taking \( p_1(t) = 0 \) and replacing \( f(c^T x) \) by \( f_2(t, x, z) = f(c^T x) - p(t) \), it remains only to verify that \( f_2 \) satisfies (2.2).

By (3.5), there is a \( k_1 \in (0, k) \) such that \( y^T f(y) - p(t) \geq (k - k_1)|y| \cdot |f(y)| \), which implies that \(-z^T K^2 f(c^T x) - p(t) \leq 0 \) for \((x, z)\) in the cone \( \{(x, z): |K^2 z - c^T x| \leq ((k - k_1)/(1 + k))|c^T x|\} \); this completes the proof of Theorem 2.

In the scalar case, instead of (3.1) one can consider a slightly more
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(3.9) \[ y^{(n)} + g_1 y^{(n-2)} + \cdots + g_{n-1} y + f(y, y', \ldots, y^{(n-1)}) = p(t), \]

where \( g_i : \mathbb{R} \to \mathbb{R}, f : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}, p : \mathbb{R} \to \mathbb{R} \) are continuous. An application of Theorem 2 gives

\[ \text{Theorem 3. Let } G_i(u) = \int_0^u g_i(s) \, ds \text { satisfy } \]

(3.10) \[ |G_i(u) - a_i u| \leq \mu |u| \quad \text{for } i = 1, \ldots, n-1, |u| \geq u_1. \]

Assume that \( a_n > 0 \) and that the polynomial \( \phi(\lambda) = \lambda^{n-1} + a_1 \lambda^{n-2} + \cdots + a_{n-1} \) has roots different from \( \lambda = (2\pi i/\omega)p, p = 1, 2, \ldots \). Let

(3.11) \[ 0 < yf(y, x) < \mu y^2 \quad \text{for } |y| \geq y_1, z \in \mathbb{R}^{n-1}. \]

Let \( p \) be \( \omega \)-periodic and either

(3.12) \[ \int_0^\omega p(s) \, ds = 0 \]

or

(3.13) \[ |f(y, z)| \geq f_0 > \max \{|p(t)| : t \in [0, \omega]\} \quad \text{for } |y| \geq y_1, z \in \mathbb{R}^{n-1}. \]

If \( \mu \) is sufficiently small, then (3.9) has at least one \( \omega \)-periodic solution.

\[ \text{Proof. Let } x_1 = y, x_T = (x_1, \ldots, x_{n-1}) \text{ and } \]

\[ G(x) = [a_{n-1} x_1 - G_{n-1}(x_1)] + \cdots + [a_1 x_{n-1} - G_1(x_{n-1})], \]

and let \( b, c, F \) be as in Theorem 2 \((m = 1)\).

Then (3.9) may be replaced by

(3.14) \[ x' = Fx + bz + F_1(t, x, z), \quad z' = F_2(t, x, z), \]

where

\[ F_1(t, x, z) = b \int_0^t p(s) \, ds + bG(x), \]

\[ F_2(t, x, z) = -f(c^T x; Fx + bz + F_1(t, x, z)) \]

if \( p \) satisfies (3.13), and

\[ F_1(t, x, z) = bG(x), \quad F_2(t, x, z) = -f(c^T x; Fx + bz + F_1(t, x, z)) + p(t) \]

otherwise. It is easily seen that the proof repeats arguments used above, hence it is left to the reader.

Theorem 3 generalizes results of [5] and partially of [6].
REFERENCES


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