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PROJECTIVE MAXIMAL RIGHT IDEALS OF SELF-INJECTIVE RINGS

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ABSTRACT. It is proved that a projective maximal right ideal M of a self-injective ring R is of the form $M = eR + J(R)$. It is also shown that if every maximal right ideal of a self-injective ring R is projective, then R must be Artin semisimple.

A ring R is called self-injective if R is injective as a right R-module. By a regular ring we mean regular in the sense of Von Neumann [4].

Let M be a right R-module; a submodule P of M is said to be essential if $P \cap N \neq (0)$, for any submodule $0 \neq N$ of M; we denote this by $M \,{}'\!\supset P$. Then the singular submodule $Z(M)$ is defined by $Z(M) = \{x \in M \mid R \,{}'\!\supset \operatorname{Ann}(x)\}$.

It is easy to see that if R is a regular ring, then $Z(R) = 0$.

It is also well known that if R is a self-injective ring with the Jacobson radical $J = J(R)$, then $J = Z(R)$ and R/J is a self-injective regular ring, and finitely many orthogonal idempotents of R/J can be lifted to orthogonal idempotents of R (see [2]).

Throughout this paper, R will denote an associative ring which does have a unity. The reader is referred to [2] for basic results on semiperfect rings.

Proposition 1. *Let R be a self-injective regular ring; then a maximal right ideal which is projective is a direct summand.*

Proof. Let M be a maximal right ideal which is projective as a right R-module; then $M = \sum_{i \in I} \bigoplus e_i R$; $e_i^2 = e_i \forall i \in I$ (see [1]). Since every maximal right ideal is either essential or a direct summand, we must assume that M is essential and, therefore, I is infinite.

Without loss of generality we can assume $e_i e_j = 0$, $i \neq j$; for if $p_i\colon M \to e_i R$ and $k_i\colon e_i R \to M$ are the natural projection and injection, respectively,

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and $\alpha_i = k_i \rho_i$, then $\alpha_i \in \mathrm{Hom}_R(M, M)$ and $\alpha_i^2 = \alpha_i$ $\forall i \in I$ and $\alpha_i \alpha_j = 0$, $i \neq j$. Clearly $a = \sum_i \alpha_i a$ $\forall a \in M$, where $\alpha_i a = 0$ for almost all $i \in I$. So $M = \sum \bigoplus \alpha_i M$ and, as R is injective, there exists for each α_i in $\mathrm{Hom}_R(M, M)$ an element $e_i' \in R$ such that $\alpha_i m = e_i' m$ $\forall m \in M$. Then since $Z(R) = 0$, we get $e_i' = e_i'^2$ $\forall i \in I$ and $e_i' e_j' = 0$, $i \neq j$.

Now we assume $M = \sum_{i \in I} \bigoplus e_i R$, $e_i^2 = e_i$, $e_i e_j = 0$, $i \neq j$. Since I is infinite we can have $I = I_1 \cup I_2$, where I_1 and I_2 are both infinite subsets of I. Set $M_1 = \sum_{i \in I_1} \bigoplus e_i R$, $M_2 = \sum_{j \in I_2} \bigoplus e_j R$.

Now if $p_i: M \to M_i$, $i = 1, 2$, is the natural projection, there exists a_i, $i = 1, 2$, in R such that $p_i(m) = a_i m$, $i = 1, 2$, $\forall m \in M$. Clearly $a_1 a_2 M = a_2 a_1 M = 0$, so $a_1 a_2 = a_2 a_1 = 0$. We claim that either a_1 or a_2 belongs to M, for if $a_2 \notin M$, then $a_2 R + M = R$ implies $1 = a_2 r + m$, $r \in R$, $m \in M$; then multiplying by a_1 we get $a_1 = a_1 a_2 r + a_1 m \in M$. So assume $a_1 \in M$, then $a_1 = m_1 + m_2$, $m_i \in M_i$, $i = 1, 2$. Multiplying by an element $x \in M_1$ we get $a_1 x = m_1 x + m_2 x$ which implies $x = m_1 x$, $m_2 x = 0$; in particular, $m_1 = m_1^2$. This shows that $M_1 = m_1 R$ and so $m_1 e_i = e_i$ $\forall i \in I_1$.

If $m_1 = \sum_{i \in A} e_i r_i$, where A is a finite subset of I_1, then $e_i m_1 = 0$ $\forall i \in I_1 - A$, and also $e_i = e_i^2 = (m_1 e_i)^2 = 0$ $\forall i \in I_1 - A$, a contradiction. Hence I must be finite.

The following result investigates the projective maximal right ideals in a self-injective ring.

Corollary 1. *If R is a self-injective ring and M a projective maximal right ideal, then $M = eR + J$, where $e = e^2$.*

Proof. By using the dual basis lemma and the fact that R is injective, it is easy to see that a right ideal M is projective if and only if there exists a collection $\{m_i\} \subset M$ such that for all $m \in M$, $m_i m = 0$ for almost all i and $m = \sum m_i m$; hence if M is a maximal right ideal, M/J is a projective maximal right ideal of R/J and, in view of Proposition 1 and the fact that idempotents lift, we have $M/J = (eR + J)/J$, $e = e^2$ which implies $M = eR + J$.

The following lemma is well known (see [2, p. 67]).

Lemma 1. *If every maximal right ideal of a ring R is a direct summand, then R is Artin semisimple.*

It is trivial to see that if the Jacobson radical of a ring R is injective it must be zero. We also have the following result:

Lemma 2. *If R is a self-injective ring, then no nonzero right ideal inside the Jacobson radical is projective.*

Proof. Let $0 \neq I$ be a projective right ideal contained in J, and let $\{a_i\}$ be a collection of elements of I such that $a_i x = 0$ for almost all i and $\forall x \in I$ and also $x = \Sigma a_i x$. But $\Sigma_{a_i x \neq 0} a_i \in J$ shows that $1 - \Sigma_{a_i x \neq 0} a_i$ is a unit, so $x(1 - \Sigma_{a_i x \neq 0} a_i) = 0$ implies $x = 0$, a contradiction.

Proposition 1, together with Lemma 1, yields the following result.

Corollary 2. *There does not exist a non-Artinian self-injective regular ring in which every maximal right ideal is countably generated.*

Proof. In a regular ring, since every countably generated right ideal is a direct sum of principal right ideals, it is projective.

Osofsky has shown [3] that a self-injective ring R which is right hereditary must be Artin semisimple. We prove a generalization of this result.

Corollary 3. *If every maximal right ideal of a self-injective ring R is projective, then R is Artin semisimple.*

Proof. Since every maximal right ideal M is of the form $M = eR + J$, it is sufficient to show that $J = 0$. Clearly R/J is Artin semisimple and, as idempotents lift modulo J, R becomes a semiperfect ring. So R contains a finite orthogonal set of primitive idempotents $\{e_1, e_2, \cdots, e_n\}$ such that $1 = e_1 + e_2 + \cdots + e_n$, and it is well known that for each i, $e_i R/e_i J$ is a simple R-module. Therefore for each i, $e_i J \oplus \Sigma_{i \neq k} \bigoplus e_k R$ is a maximal right ideal, so $J = e_1 J \oplus \cdots \oplus e_n J$ is projective, which implies $J = 0$.

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