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THE CRITICAL SET OF THE REDUCED NORM, AS AN ALGEBRAIC SET

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ABSTRACT. For an associative algebra A over k, the reduced norm $\nu: A \to k$ is a polynomial function. As such the critical set C_ν is defined, consisting of points where the differential $(d\nu)_z$ vanishes. The k-rational points of C_ν have been determined by the author under certain separability conditions (*Rational critical points of the reduced norm of an algebra*, Bull. Amer. Math. Soc. **80** (1974), 138–141); here the dimension of C_ν as an algebraic k-set is discussed.

1. **Introduction.** Let A be an associative algebra of dimension n over an infinite field k with reduced norm $\nu: A \longrightarrow k$. Then the set $(C_\nu)_k$ of its critical points lying in A is defined; it has been determined by the author [4] subject to certain separability conditions. Since ν is a k-polynomial, we may consider $\nu \in \Omega[x]$, where $x = (x_1, \cdots, x_n)$ and Ω is a universal domain for k. Then the set of critical points $C_\nu = \{z \in \Omega^n | (d\nu)_z = 0\}$ is an algebraic k-set in Ω^n. In this paper the dimension of C_ν is determined. This quantity is involved in the theory of Gauss transforms and zeta functions [5].

The heart of the proceedings is the discovery of the dimension in the case where A is a full matrix algebra over k. While this may be done by exhibiting a generic point, the preferable approach is one which makes use of a group action on C_ν; this action holds promise for giving good insights into further structure of C_ν. From this particular case, one proceeds to central simple algebras and then to special cases of simple and semisimple algebras. Then the other simple, semisimple, and abstract associative algebras are dealt with.

For basic definitions and structure theorems of algebras, see [1], [3]; for geometric concepts see [7]; the definition of critical sets is as in [5]. For a subset S of the ambient space Ω^n, we denote the set of its k-rational

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points by S_k. The following notation, as in [4], has been adopted for a simple algebra A over k: K is the center of A, so that $A = M_m(D)$, a full matrix algebra over the K-central division algebra D; the dimension of D as a vector space over K is d^2, and $[K: k] = t$; thus d is the index of A and D over K; $r = md$, the degree of A over K; and $n = r^2 t$, the (vector space) dimension of A over k. We also have the reduced norms $\nu: A \to k$, $\nu': A \to K$, $\nu^*: D \to k$, and $\nu'': K \to k$. For the general k-algebra A, we let N denote its radical, and $A_1, \cdots, A_s$, the simple component summands of A/N, each with the appropriate invariants $K_i, D_i, m_i, d_i, r_i, t_i, n_i$ and reduced norms $\bar{\nu}: A/N \to k$ and $\nu_i: A_i \to k$, $i = 1, \cdots, s$. Finally, $(d\nu)_z$ is the differential map of the norm at z.

The main theorems on k-rational points of C_ν are the following [4]:

Theorem 1. *Let A be a simple algebra over k, with n not divisible by the characteristic of k. Then*

$$(C_\nu)_k = \{z \in A \mid \mathrm{rk}(z) \le \rho(A)\},$$

where $\mathrm{rk}(z)$ *is the left row-rank of z as a matrix in* $M_m(D)$, *and* $\rho(A) = m - 1$ *or* $m - 2$, *according as* $dt = 1$ *or* $dt > 1$, *respectively.*

Theorem 2. *Let A be an associative algebra over k with unity such that no n_i is divisible by the characteristic of k. For $z \in A$, let $z + N = z_1 + \cdots + z_s$, with $z_i \in A_i$, $i = 1, \cdots, s$. Then $z \in (C_\nu)_k$ if and only if one of these conditions holds*:

(I) *There exist $i \neq j$ with* $\mathrm{rk}(z_i) \le m_i - 1$ *and* $\mathrm{rk}(z_j) \le m_j - 1$.

(II) *There exists i with* $\mathrm{rk}(z_i) \le \rho(A_i)$.

2. **A group action on** A. Let $GL_1(A)$ be the group of units of an associative algebra with unity, and $G_k = GL_1(A) \times GL_1(A)$. The action of G_k on A is then defined as follows: for $(P, Q) \in G_k$, $z \in A$, $(P, Q)z = PzQ^{-1}$. This action, of course, is compatible with the group structure of G_k.

It is not difficult to investigate this action in more general cases, but here it is sufficient to determine the orbits in the case where A is a simple algebra $A = M_m(D)$. Then the orbits can be seen to be the sets $H_k^\mu = \{z \in A \mid \mathrm{rk}(z) = \mu\}$, $\mu = 0, \cdots, m$. For using linear algebraic techniques over division rings in the spirit of [2], one can show that the set $\{1_\mu \mid \mu = 0, \cdots, m\}$ is a full set of representatives of the orbits. Here 1_μ is the diagonal matrix in $M_m(D)$ whose first μ entries are 1's and the remaining $m - \mu$ entries are 0's.

Next the stabilizer of 1_μ should be determined, i.e., $G_k^\mu = \{(P, Q) \in G_k \mid P \cdot 1_\mu \cdot Q^{-1} = 1_\mu\}$. This turns out to be the following:

$$G_k^\mu = \left\{ (P, Q) \in G_k \middle| P = \left[\begin{array}{c|c} P_1 & * \\ \hline 0 & P_2 \end{array}\right], Q = \left[\begin{array}{c|c} P_1^{-1} & 0 \\ \hline * & P_3 \end{array}\right], \right.$$
$$\left. P_1 \in GL_\mu(D),\ P_2,\ P_3 \in GL_{m-\mu}(D) \right\}.$$

Now this action provides a decomposition of $(C_\nu)_k$, under the separability hypothesis of Theorem 1. For that theorem can be phrased in terms of the action, namely, $(C_\nu)_k = H_k^0 \cup H_k^1 \cup \cdots \cup H_k^{\rho(A)}$.

Finally, we remark that the preceding treatment can be imitated in the ambient situation. That is, the group $G = GL_m(\Omega) \times GL_m(\Omega)$ acts on $M_m(\Omega)$ by the same formula as above. The orbits have the form $H^\mu = \{z \in M_m(\Omega) | \operatorname{rk}(z) = \mu\}$, $\mu = 0, \cdots, m$. And the typical element of the "full" stabilizer G^μ of 1_μ has the same format as that of G_k^μ, except that P_1 ranges through $GL_\mu(\Omega)$ and P_2 and P_3 through $GL_{m-\mu}(\Omega)$.

3. **Dimension.** First let us assume that A is a central simple algebra over k, and that its Brauer class is the identity; in other words, $t = d = 1$, $A = M_m(k)$. As usual, we also have that $\operatorname{char}(k)$ does not divide n. Then $\nu(X) \in k[X] \subseteq \Omega[X]$, and of course $\nu(X) = \det(X)$, $X = (X_1, \cdots, X_{m^2})$, and for $z \in \Omega^{m^2}$, we have $(d\nu)_z$ given by the appropriate cofactors. Therefore it is clear that in this case $C_\nu = \emptyset$ or $\{z \in M_m(\Omega) | \operatorname{rk}(z) \leq m - 2\}$, according as $m = 1$ or $m > 1$, respectively. For the nontrivial case, then, $C_\nu = H^0 \cup H^1 \cup \cdots \cup H^{m-2}$. Hence $\{H^\mu | \mu = 0, 1, \cdots, m - 2\}$ is the set of irreducible components of C_ν, since $H^\mu = G/G^\mu$.

The dimension of H^μ as a variety can now be computed from that of G^μ. From the generic point of G^μ given above, the latter is seen to be $\mu^2 + 2m^2 - 2m\mu$. It follows that $\dim(H^\mu) = 2m\mu - \mu^2$, and $\dim(C_\nu) = \max\{\dim(H^\mu) | \mu = 0, \cdots, m - 2\} = m^2 - 4 = n - 4$. This is a special case of

Theorem 3. *Let A be a central simple algebra over k of dimension n, with $n > 1$ and not divisible by* $\operatorname{char}(k)$. *Then* $\dim(C_\nu) = n - 4$.

To complete the proof, first let L be a maximal (hence splitting) subfield of A. Then $[L: k] = r$, where $r^2 = n$. Now let the map $F: A \to M_r(L)$ be the right regular representation of A on itself as an L-space, for a fixed L-basis of A. Since F is k-linear, $F \in \mathfrak{M}_L$ in the sense of [6].

Moreover, the multiplicative property of F implies, under the terms of [6, Proposition 6, p. 168], that $\nu = \delta \circ F$, where $\delta\colon M_r(L) \to L$ is the determinant map and the codomain of ν is considered to be L.

Note that F is an embedding, because of the simplicity of A. Further, its k-linearity implies that for $z \in A$, $F(z) = (F_{ij}(z))$, with F_{ij} a homogeneous linear polynomial function in the k-coordinates of z, with coefficients in L. As such, it extends to the ambient space, along with ν and δ. In other words, the commutative diagram is now

$$\begin{tikzcd} \Omega^n \arrow[r, "F"] \arrow[dr, "\nu"'] & \Omega^n \arrow[d, "\delta"] \\ & \Omega \end{tikzcd}$$

all maps being defined over L. Then for $z \in \Omega^n, (d\nu)_z = (d\delta)_{F(z)} \circ (dF)_z$. Now since $(dF)_z$ is constant and invertible, F is an isomorphism (over L) between C_ν and C_δ. But $\dim(C_\delta) = n - 4$, according to the special case treated above; thus $\dim(C_\nu) = n - 4$, and the theorem has been proved.

Next, the case is treated where A is simple of index 1; since $t = 1$ has already been treated, we may assume $t > 1$. In other words, $A = M_m(K)$, $[K\colon k] > 1$. Then [4] $\nu = \nu'' \circ \nu' = \prod(\nu')^\sigma$, where σ runs through the t distinct k-embeddings of K in an algebraic closure. Thus

$$(d\nu)_z = \sum_\sigma (d(\nu')^\sigma)_z \prod_{\tau \neq \sigma} (\nu' z)^\tau.$$

Therefore, by the Euler identity for homogeneous polynomials, for $z \in C_\nu$, there is a σ with $z \in V_\sigma$, where $V_\sigma = \{z \in \Omega^n \mid (\nu' z)^\sigma = 0\}$ for $(\nu')^\sigma \in \bar{k}[X_1, \cdots, X_n] \subset \Omega[X_1, \cdots, X_n]$. Thus for some σ we have $(d\nu)_z = (d(\nu')^\sigma)_z \prod_{\tau \neq \sigma} (\nu' z)^\tau$. This equality, since it is a polynomial identity, shows that for all $\sigma \neq \tau$, the intersection $V_\sigma \cap V_\tau$ is an algebraic subset of C. In fact the equality yields a decomposition of C_ν into algebraic subsets:

$$\Psi C_\nu = \bigcup_\sigma \left(C_\sigma \cup \left(\bigcup_{\sigma \neq \tau} (V_\sigma \cap V_\tau) \right) \right),$$

where C_σ is the critical set for $(\nu')^\sigma$.

This decomposition allows us to conclude that $\dim(C_\nu) = n - 2$ in this case. To do this, note that it is sufficient to show that the maximum dimension attained by the components of $V_\sigma \cap V_\tau$, for all $\sigma \neq \tau$, is $n - 2$; for C_σ, as a proper subvariety of V_σ, cannot have dimension exceeding $n - 2$. However, any component of $V_\sigma \cap V_\tau$ can similarly have dimension at most $n-2$. On the other hand, the varieties V_σ are distinct, because of the separability hypothesis, and their defining polynomials are absolutely irreducible. Therefore, according to intersection theory for homogeneous polynomials, $\dim(V_\sigma \cap V_\tau) \geq n-2$.

At this point we have examined simple algebras of special types: $t=1$ or $d=1$. Rather than turn to the "general" simple algebra, we now treat the semisimple algebra whose components are of the special types studied. In the following discussion, then, the semisimple algebra A has components $A_1, \cdots, A_s$, as described earlier, with the restrictions $\min(t_i, d_i) = 1$, and n_i is not divisible by the characteristic of k, for $i = 1, \cdots, s$. Furthermore, assume $s > 1$.

In this situation we claim $\dim(C_\nu) = n-2$. For let $z \in A$, $z = z_1 + \cdots + z_s$, $z_i \in A_i$, and define $\bar{\nu}_i(z) = \nu_i(z_i)$. Then since $\operatorname{codim}(C_{\nu_i}) = \operatorname{codim}(C_{\bar{\nu}_i})$, we have $\dim(C_{\bar{\nu}_i}) = n-4$ or $n-2$, according to previous discussions. Now $\nu(z) = \Pi\, \nu_i(z_i)$ [4], so that as above $(d\nu)_z = (d\bar{\nu}_i)_z \,\Pi_{j\neq i}\, \bar{\nu}^j(z)$ for some i. This yields the decomposition

$$C_\nu = \bigcup_i \left(C_{\bar{\nu}_i} \cup \left(\bigcup_{j \neq i} (V_i \cap V_j) \right) \right),$$

where $V_i = \{z \in \Omega^n \mid \nu_i(z_i) = 0\}$. But clearly $\operatorname{codim}(V_i \cap V_j) = \operatorname{codim}(V_i) + \operatorname{codim}(V_j)$. Therefore $\dim(V_i \cap V_j) = n-2$, and so $\dim(C_\nu) = n-2$.

We may now state the main result for simple algebras.

Theorem 4. *Let A be a simple algebra over k with dimension n, $n > 1$ and not divisible by the characteristic of k. Then* $\dim(C_\nu) = n-4$ *or* $n-2$, *according as* $t=1$ *or* $t>1$, *respectively.*

Since the case $t=1$ was the content of Theorem 3, it remains to deal with $t \neq 1$. Let L be a maximal subfield of A. Then $[A: K] = r^2$ and $[L: K] = r$. Consider $\Psi: A \to A \otimes_k L$, where the latter is identified with $\bigoplus_{i=1}^t (M_r(L))$, defined by $\Psi(z) = \bigoplus F(z)^\sigma$, with σ running through the t distinct k-embeddings of K in $\bar{k}$, extended to L, and F as defined above. Then if $\delta': A \otimes_k L \to L$ is the reduced norm, we have, extending the earlier discussion, $\delta' \circ \Psi = \nu$. Furthermore, Ψ is an L-polynomial mapping and so may be viewed as $\Psi: \Omega^n \to \Omega^n$. Hence, in an analogous way, Ψ may be seen to be an L-isomorphism between C_ν and $C_{\delta'}$. But since $A \otimes_k L$ is a semisimple algebra of the type treated, we have $\dim(C_{\delta'}) = n-2$. Thus $\dim(C_\nu) = n-2$.

With this result for simple algebras, we may state and prove the dimension theorem for a general case.

Theorem 5. *Let A be an associative k-algebra of dimension $n \geq 1$, with $A/N = A_1 \oplus \cdots \oplus A_s$ a decomposition into simple components with invariants described above, such that no n_i is divisible by the characteristic of k. Then if* $\dim_k A/N = 1$, $C_\nu = \emptyset$. *Otherwise* $\dim(C_\nu) = n - 4$ *or* $n - 2$, *according as* $st_1 \cdots t_s = 1$ *or otherwise.*

To prove this, we first note that for $A/N \approx k$ the claim is trivial. In the other situation, we may reduce to the semisimple case, since [4] $\operatorname{codim}(C_\nu) = \operatorname{codim}(C_{\bar{\nu}})$. But for semisimple algebras A, the cases $st_1 \cdots t_s = 1$ and $s = 1$ are included in previous theorems.

Let us therefore assume that $A = A_1 \oplus \cdots \oplus A_s$, with $s > 1$. Then the proof is very similar to the discussion before Theorem 4 of the "special" type of semisimple algebras; the only difference is that Theorem 5 now allows us to lift the earlier restriction on the components $A_1, \cdots, A_s$.

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