PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 2, April 1975

ON THE TRIVIALITY OF HOMOGENEOUS ALGEBRAS OVER AN ALGEBRAICALLY CLOSED FIELD[1]

LOWELL SWEET

ABSTRACT. Let A be a finite-dimensional algebra (not necessarily associative) over a field K. Then A is said to be homogeneous if $\mathrm{Aut}(A)$ acts transitively on the one-dimensional subspaces of A. If A is homogeneous and K is algebraically closed, then it is shown that either $A^2 = 0$ or $\dim A = 1$.

All algebras are assumed to be finite dimensional and not necessarily associative. Let A be an algebra over a field K and let $\mathrm{Aut}(A)$ denote the group of algebra automorphisms of A. We say that A is extremely homogeneous if $\mathrm{Aut}(A)$ acts transitively on $A \setminus \{0\}$. Extremely homogeneous algebras have been investigated by Kostrikin [4]. We say that A is homogeneous if $\mathrm{Aut}(A)$ acts transitively on the one-dimensional subspaces of A. Homogeneous algebras over finite fields have been investigated by Shult [5], [6] and Gross [3]. Swierczkowski classified all real homogeneous Lie algebras [7], and finally Djoković classified all real homogeneous algebras [2]. A homogeneous algebra A is said to be nontrivial if $A^2 \neq 0$ and $\dim A > 1$. The purpose of this paper is to show that there are no nontrivial homogeneous algebras over an algebraically closed field.

Let A be an arbitrary algebra over any field K. Then left multiplication by a fixed element $a \in A$ induces a linear map on A which is denoted by L_a. Similarly right multiplication by a induces a linear map denoted by R_a. If a basis of A is chosen, we do not distinguish between the operator L_a and its matrix representation relative to this fixed basis. By $\mathrm{End}(A)$ we indicate the vector space of linear maps on A, and by L we indicate the subspace of $\mathrm{End}(A)$ consisting of all L_x as x runs through A, and similarly for R.

Now let A be a homogeneous algebra over an arbitrary field K. If

Received by the editors January 24, 1974.

AMS (MOS) subject classifications (1970). Primary 17A99; Secondary 20F99, 15A03.

Key words and phrases. Homogeneous algebra, algebraically closed field, special nil algebra, Jordan normal form.

[1] This work was supported in part by NRC Grant A9119.

$a, b \in A \setminus \{0\}$, then the homogeneity condition implies that L_a and L_b are projectively similar, and similarly for R_a and R_b. It is also easy to show that $x \longrightarrow L_x$ is a linear isomorphism of $A \longrightarrow L$.

Definition. An algebra A over a field K is said to be a left (right) special nil algebra if $x \in A \setminus \{0\}$ implies that L_x (R_x) is nilpotent and if $x, y \in A \setminus \{0\}$ implies that L_x and L_y (R_x and R_y) are similar. A is said to be a special nil algebra if it is both a left special nil algebra and a right special nil algebra.

Theorem 1. *Let A be a homogeneous algebra of* $\dim n > 1$ *over an algebraically closed field K. Then A is a special nil algebra.*

Proof. The proof is a generalization of a theorem of Boen, Rothaus and Thompson [1]. Choose a basis $\{e_1, e_2, \cdots, e_n\}$ for A. Let $a = \sum_{i=1}^n \lambda_i e_i$ and suppose the characteristic polynomial of L_a is $X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n$. Now $L_a = \sum_{i=1}^n \lambda_i L_{e_i}$ and so the elements of L_a are linear functions in the variables $\lambda_1, \lambda_2, \cdots, \lambda_n$. Let i be a positive integer such that $1 \leq i \leq n$. Since a_i is the sum up to the sign of the principal $i \times i$ subdeterminants of L_a, it follows that a_i is a homogeneous polynomial of degree i in the variables $\lambda_1, \lambda_2, \cdots, \lambda_n$. But since K is algebraically closed and $\dim A = n > 1$, it follows that there exists a nonzero n-tuple $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ such that $a_i(\lambda_1, \lambda_2, \cdots, \lambda_n) = 0$. Now let b be any nonzero element of A. Since L_b and L_a are projectively similar, it follows that if b_i is the corresponding coefficient of the characteristic polynomial of L_b, then $b_i = \lambda^i a_i = 0$ for some $\lambda \in K$. But since i was any integer in the set $\{1, 2, \cdots, n\}$, it follows that the characteristic polynomial of L_b must be X^n and so L_b is nilpotent by the Cayley-Hamilton theorem. It follows that A is a left special nil algebra, and a similar argument shows that A is a right special nil algebra.

Definition. Let A be a special nil algebra. Since $x, y \in A \setminus \{0\}$ imply that L_x and L_y are similar, it follows that $f(x) = \operatorname{rank} L_x$, $x \in A \setminus \{0\}$ is a constant, say r, and we say that $r = \operatorname{rank} L$. Rank R is defined in a similar manner.

Definition. If A is any algebra then A^{opp} indicates the algebra obtained from A by reversing the order of multiplication. That is, in A^{opp}, $a \circ b = ba$.

Our main result now follows directly from the following theorem.

Theorem 2. *Let A be a special nil algebra of dimension n over a field K. If $n \leq$ card K then $A^2 = 0$.*

Proof. Let A be a counterexample to the above statement. If rank $L >$

rank R, we replace A by A^{opp}, which is still a special nil algebra, and $A^2 \neq 0$ implies that $(A^{\text{opp}})^2 \neq 0$. Hence, without loss of generality, we may assume that rank $L \leq$ rank R.

Let a be a fixed element of $A \setminus \{0\}$. Since L_a is nilpotent, all the eigenvalues of L_a are zero and, hence, lie in K, and it follows that a basis $\{e_1, e_2, \cdots, e_n\}$ of A can be chosen so that $L_a = Q$ is in the Jordan normal form. That is, $Q = B_1 \oplus B_2 \oplus \cdots \oplus B_{r+1}$, where all the entries of B_i $(1 \leq i \leq r)$ are zero except for the first subdiagonal which is filled with 1's, and B_{r+1} is a zero matrix. Denote the size of B_i by m_i for $1 \leq i \leq r+1$. We may assume that $m_1 \geq m_2 \geq \cdots \geq m_r$. Let $e_{m_1} = b$. Clearly $L_a(b) = 0$ and so $a \in \ker R_b$. Also $\ker R_b \neq A$ because if $Ab = 0$ then the fact that A is a right special nil algebra would imply that $A^2 = 0$. Let A_2 be any vector space complement of $\ker R_b$. Then we have $A = \ker R_b \oplus A_2$.

Since the map $\phi = A \longrightarrow L$ defined by $x \longrightarrow L_x$ is a linear isomorphism, it follows that $L = \phi(\ker R_b) \oplus \phi(A_2)$.

Let $\dim \ker R_b = k$. Then

$$n - k = \dim \phi(A_2) = \text{rank } R \geq \text{rank } L = \sum_{i=1}^{r} (m_i - 1).$$

Now let $x \in A_2 \setminus \{0\}$. Then $Q + \lambda L_x$ must be similar to Q, $\forall \lambda \in K$, and so $(Q + \lambda L_x)^{m_1} = 0, \forall \lambda \in K$. Since Q and L_x are nilpotent of index m_1, it follows that the degree in λ of the matrix polynomial $(Q + \lambda L_x)^{m_1}$ is $\leq m_1 - 1 <$ card K under the restriction $n \leq$ card K in the hypothesis. Hence every coefficient of the polynomial $(Q + \lambda L_x)^{m_1}$ must be zero, and so in particular the coefficient of λ must be zero. That is

$$B = Q^{m_1 - 1} L_x + Q^{m_1 - 2} L_x Q + \cdots + L_x Q^{m_1 - 1}$$

$$= Q(Q^{m_1 - 2} L_x + Q^{m_1 - 3} L_x Q + \cdots + L_x Q^{m_1 - 2}) + L_x Q^{m_1 - 1}$$

$$= QC + L_x Q^{m_1 - 1} = 0.$$

Consider the entries lying in the intersection of the first column of B and the rows

$$1,\ m_1 + 1,\ m_1 + m_2 + 1,\ \cdots,\ \sum_{i=1}^{r} m_i + 1,\ \sum_{i=1}^{r} m_i + 2,\ \cdots,\ n.$$

Because of the structure of Q, it is easily checked that the corresponding entries of QC are all zero and so the same must be true for the corresponding entries of $L_x Q^{m_1 - 1}$. But this implies that if $L_x = (l_{ij})$ then

$$l_{1,m_1} = l_{1+m_1,m_1} = \cdots = l_{1+\sum_{i=1}^r m_i, m_1} = l_{2+\sum_{i=1}^r m_i, m_1} = \cdots = l_{nm_1} = 0.$$

Now as a consequence of the fact that any system of $n - k - 1$ homogeneous linear equations in $n - k$ unknowns must have a nontrivial solution, it follows that it is possible to take a nontrivial linear combination of $n - k$ independent matrices to get a matrix with zeros in at least $n - k - 1$ specified positions. Hence there must exist $f \in A_2 \setminus \{0\}$ such that if $L_f = (f_{ij})$ then $f_{im_1} = 0$ whenever $i \neq m_1$.

But now L_f has eigenvalue $f_{m_1 m_1} \in K$, and so $f_{m_1 m_1} = 0$ since L_f is nilpotent. Hence $L_f(b) = 0$, which is impossible, because $f \in A_2 \cap \ker R_b = \{0\}$, and the proof is complete.

Remark. I would like to thank my supervisor, Professor D. Ž. Djoković, for suggesting this problem and for his encouragement. The author is also indebted to the National Research Council of Canada and the University of Waterloo for their financial assistance.

REFERENCES

1. J. Boen, O. Rothaus and J. Thompson, *Further results on p-automorphic p-groups*, Pacific J. Math. 12 (1962), 817–821. MR 27 #2553b.

2. D. Ž. Djoković, *Real homogeneous algebras*, Proc. Amer. Math. Soc. 41 (1973), 457–462.

3. F. Gross, *Finite automorphic algebras over* GF(2), Proc. Amer. Math. Soc. 31 (1972), 10–14. MR 44 #4063.

4. A. I. Kostrikin, *On homogeneous algebras*, Izv. Akad. Nauk SSSR Ser. Mat. 29 (1965), 471–484. (Russian) MR 31 #219.

5. E. E. Shult, *On finite automorphic algebras*, Illinois J. Math. 13 (1969), 625–653. MR 40 #1441.

6. ———, *On the triviality of finite automorphic algebras*, Illinois J. Math. 13 (1969), 654–659. MR 40 #1442.

7. S. Swierczkowski, *Homogeneous Lie algebras*, Bull. Austral. Math. Soc. 4 (1971), 349–353. MR 43 #6277.

UNIVERSITY OF PRINCE EDWARD ISLAND, CHARLOTTETOWN, PRINCE EDWARD ISLAND, CANADA