ON THE TRIVIALITY OF HOMOGENEOUS ALGEBRAS
OVER AN ALGEBRAICALLY CLOSED FIELD\(^1\)

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ABSTRACT. Let \( A \) be a finite-dimensional algebra (not necessarily associative) over a field \( K \). Then \( A \) is said to be homogeneous if \( \text{Aut}(A) \) acts transitively on the one-dimensional subspaces of \( A \). If \( A \) is homogeneous and \( K \) is algebraically closed, then it is shown that either \( A^2 = 0 \) or \( \dim A = 1 \).

All algebras are assumed to be finite dimensional and not necessarily associative. Let \( A \) be an algebra over a field \( K \) and let \( \text{Aut}(A) \) denote the group of algebra automorphisms of \( A \). We say that \( A \) is extremely homogeneous if \( \text{Aut}(A) \) acts transitively on \( A \setminus \{0\} \). Extremely homogeneous algebras have been investigated by Kostrikin [4]. We say that \( A \) is homogeneous if \( \text{Aut}(A) \) acts transitively on the one-dimensional subspaces of \( A \). Homogeneous algebras over finite fields have been investigated by Shult [5], [6] and Gross [3]. Swierczkowski classified all real homogeneous Lie algebras [7], and finally Djoković classified all real homogeneous algebras [2]. A homogeneous algebra \( A \) is said to be nontrivial if \( A^2 \neq 0 \) and \( \dim A > 1 \). The purpose of this paper is to show that there are no nontrivial homogeneous algebras over an algebraically closed field.

Let \( A \) be an arbitrary algebra over any field \( K \). Then left multiplication by a fixed element \( a \in A \) induces a linear map on \( A \) which is denoted by \( L_a \). Similarly right multiplication by \( a \) induces a linear map denoted by \( R_a \). If a basis of \( A \) is chosen, we do not distinguish between the operator \( L_a \) and its matrix representation relative to this fixed basis. By \( \text{End}(A) \) we indicate the vector space of linear maps on \( A \), and by \( L \) we indicate the subspace of \( \text{End}(A) \) consisting of all \( L_x \) as \( x \) runs through \( A \), and similarly for \( R \).

Now let \( A \) be a homogeneous algebra over an arbitrary field \( K \). If

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a, b ∈ A\{0}, then the homogeneity condition implies that $L_a$ and $L_b$ are projectively similar, and similarly for $R_a$ and $R_b$. It is also easy to show that $x \to L_x$ is a linear isomorphism of $A \to L$.

Definition. An algebra $A$ over a field $K$ is said to be a left (right) special nil algebra if $x \in A\{0\}$ implies that $L_x$ ($R_x$) is nilpotent and if $x, y \in A\{0\}$ implies that $L_x$ and $L_y$ ($R_x$ and $R_y$) are similar. $A$ is said to be a special nil algebra if it is both a left special nil algebra and a right special nil algebra.

Theorem 1. Let $A$ be a homogeneous algebra of dim $n > 1$ over an algebraically closed field $K$. Then $A$ is a special nil algebra.

Proof. The proof is a generalization of a theorem of Boen, Rothaus and Thompson [1]. Choose a basis $\{e_1, e_2, \ldots, e_n\}$ for $A$. Let $a = \sum_{i=1}^n \lambda_i e_i$ and suppose the characteristic polynomial of $L_a$ is $X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n$. Now $L_a = \sum_{i=1}^n \lambda_i L_{e_i}$ and so the elements of $L_a$ are linear functions in the variables $\lambda_1, \ldots, \lambda_n$. Let $i$ be a positive integer such that $1 \leq i \leq n$. Since $a_i$ is the sum up to the sign of the principal $i \times i$ subdeterminants of $L_a$, it follows that $a_i$ is a homogeneous polynomial of degree $i$ in the variables $\lambda_1, \ldots, \lambda_n$. But since $K$ is algebraically closed and dim $A = n > 1$, it follows that there exists a nonzero $n$-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $a_i(\lambda_1, \lambda_2, \ldots, \lambda_n) = 0$. Now let $b$ be any nonzero element of $A$. Since $L_b$ and $L_a$ are projectively similar, it follows that if $b_i$ is the corresponding coefficient of the characteristic polynomial of $L_b$, then $b_i = \lambda a_i = 0$ for some $\lambda \in K$. But since $i$ was any integer in the set $\{1, 2, \ldots, n\}$, it follows that the characteristic polynomial of $L_b$ must be $X^n$ and so $L_b$ is nilpotent by the Cayley-Hamilton theorem. It follows that $A$ is a left special nil algebra, and a similar argument shows that $A$ is a right special nil algebra.

Definition. Let $A$ be a special nil algebra. Since $x, y \in A\{0\}$ imply that $L_x$ and $L_y$ are similar, it follows that $f(x) = \text{rank } L_x, x \in A\{0\}$ is a constant, say $r$, and we say that $r = \text{rank } L$. Rank $R$ is defined in a similar manner.

Definition. If $A$ is any algebra then $A^{\text{opp}}$ indicates the algebra obtained from $A$ by reversing the order of multiplication. That is, in $A^{\text{opp}}$, $a \circ b = ba$.

Our main result now follows directly from the following theorem.

Theorem 2. Let $A$ be a special nil algebra of dimension $n$ over a field $K$. If $n \leq \text{card } K$ then $A^2 = 0$.

Proof. Let $A$ be a counterexample to the above statement. If rank $L >
rank $R$, we replace $A$ by $A^{\text{opp}}$, which is still a special nil algebra, and $A^2 \neq 0$ implies that $(A^{\text{opp}})^2 \neq 0$. Hence, without loss of generality, we may assume that rank $L \leq$ rank $R$.

Let $a$ be a fixed element of $A \setminus \{0\}$. Since $L_a$ is nilpotent, all the eigenvalues of $L_a$ are zero and, hence, lie in $K$, and it follows that a basis

$\{e_1, e_2, \ldots, e_n\}$ of $A$ can be chosen so that $L_a = Q$ is in the Jordan normal form. That is, $Q = B_1 \oplus B_2 \oplus \cdots \oplus B_{r+1}$, where all the entries of $B_i$ (for $1 \leq i \leq r$) are zero except for the first subdiagonal which is filled with 1's, and $B_{r+1}$ is a zero matrix. Denote the size of $B_i$ by $m_i$ for $1 \leq i \leq r+1$.

We may assume that $m_1 \geq m_2 \geq \cdots \geq m_r$. Let $e_{\frac{m_1}{2}} = b$. Clearly $L_a(b) = 0$ and so $a \in \ker R_b$. Also ker $R_b \neq A$ because if $Ab = 0$ then the fact that $A$ is a right special nil algebra would imply that $A^2 = 0$. Let $A_2$ be any vector space complement of ker $R_b$. Then we have $A = \ker R_b \oplus A_2$.

Since the map $\phi = A \rightarrow L$ defined by $x \rightarrow L_x$ is a linear isomorphism, it follows that $L = \phi(\ker R_b) \oplus \phi(A_2)$.

Let dim ker $R_b = k$. Then

$$n - k = \dim \phi(A_2) = \text{rank } R \geq \text{rank } L = \sum_{i=1}^{r} (m_i - 1).$$

Now let $x \in A_2 \setminus \{0\}$. Then $Q + \lambda L_x$ must be similar to $Q$, $\forall \lambda \in K$, and so $(Q + \lambda L_x)^{m_1} = 0$, $\forall \lambda \in K$. Since $Q$ and $L_x$ are nilpotent of index $m_1$, it follows that the degree in $\lambda$ of the matrix polynomial $(Q + \lambda L_x)^{m_1}$ is $\leq m_1 - 1 < \text{card } K$ under the restriction $n \leq \text{card } K$ in the hypothesis. Hence every coefficient of the polynomial $(Q + \lambda L_x)^{m_1}$ must be zero, and so in particular the coefficient of $\lambda$ must be zero. That is

$$B = Q^{m_1 - 1} L_x + Q^{m_1 - 2} L_x Q + \cdots + L_x Q^{m_1 - 1} = Q(Q^{m_1 - 2} L_x + Q^{m_1 - 3} L_x Q + \cdots + L_x Q^{m_1 - 2}) + L_x Q^{m_1 - 1} = QC + L_x Q^{m_1 - 1} = 0.$$

Consider the entries lying in the intersection of the first column of $B$ and the rows

$$1, m_1 + 1, m_1 + m_2 + 1, \cdots, \sum_{i=1}^{r} m_i + 1, \sum_{i=1}^{r} m_i, 2, \cdots, n.$$

Because of the structure of $Q$, it is easily checked that the corresponding
entries of $QC$ are all zero and so the same must be true for the corresponding entries of $L_xQ^{m_1-1}$. But this implies that if $L_x = (l_{ij})$ then

$$l_{i,m_1} = l_{i+1,m_1} = \cdots = l_{i+\sum_{i=1}^{r}m_i,m_1} = l_{i+2\sum_{i=1}^{r}m_i,m_1} = \cdots = l_{nm_1} = 0.$$  

Now as a consequence of the fact that any system of $n - k - 1$ homogeneous linear equations in $n - k$ unknowns must have a nontrivial solution, it follows that it is possible to take a nontrivial linear combination of $n - k$ independent matrices to get a matrix with zeros in at least $n - k - 1$ specified positions. Hence there must exist $f \in A_2 \setminus \{0\}$ such that if $L_f = (f_{ij})$ then $f_{im_1} = 0$ whenever $i \neq m_1$.

But now $L_f$ has eigenvalue $f_{m_1,m_1} \in K$, and so $f_{m_1,m_1} = 0$ since $L_f$ is nilpotent. Hence $L_f(b) = 0$, which is impossible, because $f \in A_2 \cap \ker R_b = \{0\}$, and the proof is complete.

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