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A NOTE ON A COROLLARY OF SARD'S THEOREM

JOHN C. WELLS

ABSTRACT. A corollary of Sard's theorem is the following:

Corollary. *Let $f: K \to R^n$ be a smooth (i.e. $f \in C^k$, $k \geq 1$) map defined on a compact subset K of R^n. Let $C = \{y | f^{-1}(y)$ is infinite$\}$. Then the Lebesgue measure of C is zero.*

The purpose of this note is to show that a similar version of this theorem holds for Lipschitz functions.

Theorem. *Let $f: K \to R^n$ with f Lipschitz, K a measurable subset of R^n and $m(K)$ the Lebesgue measure of K, less than ∞. Let $C = \{y | f^{-1}(y)$ is infinite$\}$. Then $m(C) = 0$.*

The proof of the theorem will require two lemmas.

Lemma 1. *Let $f: K \to R^m$ with K a compact subset of R^n and f a continuous function. Then there exists $K' \subset K$ with $f(K') = f(K)$, $f|_{K'}$ one to one and K' a Borel set.*

Proof. Let x_i denote the ith coordinate of x. Define for $j = 1, 2, 3, \cdots$, $A_j = \{x | x \in K$ and there exists $y \in K$ with $y_1 \leq x_1 - 1/j$ and $f(y) = f(x)$ or $y \in K$ with $y_1 = x_1$, $y_2 \leq x_2 - 1/j$ and $f(y) = f(x)$ or $\cdots$ or $y \in K$ with $y_1 = x_1, \cdots, y_{n-1} = x_{n-1}$, $y_n \leq x_n - 1/j$ and $f(y) = f(x)\}$. Since A_j is compact for each j, $K' = K - \bigcup_j A_j$ is Borel. If $x \in f(K)$ observe that y defined by

$$y_1 = \operatorname{Inf}\{z_1 | f(z) = f(x)\}, \cdots, y_n = \operatorname{Inf}\{z_n | z_1 = y_1, \cdots, z_{n-1} = y_{n-1}, f(z) = f(x)\}$$

is not contained in A_j for any j and also $f(y) = f(x)$. Thus $f(K') = f(K)$. Clearly $f|_{K'}$ is one to one.

Lemma 2. *Let $f: K \to R^m$ be a continuous map from K a measurable subset of R^n. Then there exist K', K'' with $K'' \subset K' \subset K$, K'' and K' Borel sets, $f|_{K''}$ one to one, $m(K - K') = 0$ and $f(K'') = f(K')$.*

Proof. There exists an F_σ set K', with $K' \subset K$ and $m(K - K') = 0$, and we can write $K' = \bigcup_n K_n$ where K_n is compact for each n. By Lemma 1

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there are Borel sets $K'_n \subset K_n$ with $f(K'_n) = f(K_n)$ and $f|_{K'_n}$ one to one for each n. Define recursively $D_1 = K'_1, \cdots, D_k = D_{k-1} \cup (K'_k - f^{-1}(f(D_{k-1})))$. Then $K'' = \bigcup_k D_k$ is Borel, $f(K'') = f(K')$ and $f|_{K''}$ is one to one.

Proof of Theorem. Suppose $m(C) = a > 0$. Let $A_1 = f^{-1}(C)$ and find by Lemma 2 $N''_1 \subset N'_1 \subset A_1$ with N''_1 and N'_1 Borel, $m(A_1 - N'_1) = 0$, $f|_{N''_1}$ one to one and $f(N''_1) = f(N'_1)$. Since f is Lipschitz, $m(f(A_1 - N'_1)) = 0$. Therefore $m(f(N''_1)) = a$. Now $f(A_1 - N''_1) = C$ by the definition of C. Thus for $k = 2, 3, \cdots$ by letting $A_k = A_{k-1} - N''_{k-1}$ we can repeat the above argument to find $N''_k \subset N'_k \subset A_k$ with $f|_{N''_k}$ one to one, $f(N''_k) = f(N'_k)$ and $m(f(N''_k)) = a$. Suppose that L is the Lipschitz constant for f; then $m(N''_k) \geq a/L$ for each K. But since $N''_k \cap N'_k = \varnothing$ for $k = k'$, this implies $m(K) = \infty$, a contradiction.

Remarks. The requirement that f be Lipschitz in the theorem cannot be weakened to a requirement of continuity. To show this let f be the first coordinate of a continuous map of the unit interval I onto the unit square I^2 (i.e. a space-filling curve). Then $f^{-1}(y)$ is infinite for each $y \in I$.

On the other hand, for $n = 1$ absolute continuity will suffice as the following argument shows. Suppose f is a real valued a.c. function on I. Then f is differentiable except on a set A_1 of measure zero. $m(f(A_1)) = 0$ since f is a.c. Let $A_2 = \{x | f'(x) = 0\}$ and let η be an arbitrary positive number. For each $x \in A_2$ there are arbitrarily small intervals containing x such that $|f(y) - f(x)| \leq |y - x|$ when y is in the intervals. Thus the collection $\mathcal{F}$ of all such intervals with x varying over A_2 is a Vitale covering of A_2, so there is a disjoint collection $\{I_n\} \subset \mathcal{F}'$ with $m(A_2 - \bigcup_n I_n) = 0$. But then

$$m(f(A_2)) \leq m\left(f\left(A_2 - \bigcup_n I_n\right)\right) + m\left(f\left(\bigcup_n I_n\right)\right) \leq 0 + 2\eta \cdot \sum_n \text{length}(I_n) \leq 2\eta.$$

Since η is arbitrary, $m(f(A_2)) = 0$. Finally, if $y \in f(I) - f(A_1) - f(A_2)$ then $f^{-1}(y)$ must consist of isolated points and hence be finite. Thus $m\{y | f^{-1}(y)$ is infinite$\} = 0$.

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DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, NORTHRIDGE, CALIFORNIA 91324