PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 2, April 1975

A SHORT PROOF OF AN EXISTENCE THEOREM IN CONSTRUCTIVE MEASURE THEORY

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ABSTRACT. The most important example of an integration space in the constructive measure theory of Bishop and Cheng is the couple (X, μ), where X is a locally compact metric space and μ is a nonnegative linear function on the space of continuous functions of compact support on X. Bishop and Cheng's proof that (X, μ) is indeed an integration space is rather involved. In this paper a much simpler proof is given.

Let (X, d) be a locally compact metric space and let μ be a nonnegative linear function on the space C of continuous functions on X with compact supports. To rule out trivial cases assume that $\mu(f) > 0$ for some f in C. Bishop and Cheng [1] gave a constructive proof for the following

Theorem. *If $(f_m)_{m=0}^{\infty}$ is a sequence of nonnegative functions in C with $\sum_{m=1}^{\infty}\mu(f_m)$ convergent and less than $\mu(f_0)$, then there exists an x in X with $\sum_{m=1}^{\infty} f_m(x)$ convergent and less than $f_0(x)$.*

This theorem is the hardest step in showing that (X, μ) is an integration space as defined in [1]. It has as consequences many existence theorems. The proof given in [1] is rather involved. We now give a much simpler proof.

Let K be a compact set outside which f_0 vanishes. Construct a finite $\frac{1}{2}$-net $\{y_1, \cdots, y_N\}$ of K, and functions j_n $(1 \le n \le N)$ in C with values in $[0, 1]$ and such that $j_n(x) = 1$ or 0 according as $d(x, y_n) < \frac{1}{2}$ or > 1. Define $g_1 = j_1$ and $g_n = g_{n-1} \vee j_n - g_{n-1}$ $(1 < n \le N)$. Clearly $\sum_1^N g_n \le 1$, and equality holds on K. Therefore, the hypothesis, $\sum_{m=1}^{\infty}\mu(f_m) < \mu(f_0)$, of the Theorem implies

$$\sum_{n=1}^{N}\sum_{m=1}^{\infty}\mu(f_m g_n) < \sum_{n=1}^{N}\mu(f_0 g_n).$$

Hence there exists some n $(1 \le n \le N)$ such that $\sum_{m=1}^{\infty}\mu(f_m g_n) < \mu(f_0 g_n)$.

Received by the editors March 5, 1974.
AMS (MOS) subject classifications (1970). Primary 02E99, 28A30.
Key words and phrases. Constructive analysis, Daniell integrals.
[1]Research partially supported by NSF Grant GP-29233.

The function $h_1 \equiv g_n$ vanishes outside a compact set of diameter at most 1. Applying a similar argument to the last inequality, we construct inductively a sequence (h_k) of nonnegative functions in C where h_k is supported by a compact set of diameter at most k^{-1}, and such that

$$\sum_{m=1}^{\infty} \mu(f_m h_1 \cdots h_k) < \mu(f_0 h_1 \cdots h_k) \qquad (k \geq 1).$$

We next observe that if $f \in C$ is such that $\mu(f) > 0$ then there exists $x \in X$ with $f(x) > 0$. This observation together with $\Sigma_{m=1}^{k} \mu(f_m h_1 \cdots h_k) < \mu(f_0 h_1 \cdots h_k)$ immediately yields for each k some $x_k \in X$ with

$$0 \leq \sum_{m=1}^{k} f_m h_1 \cdots h_k(x_k) < f_0 h_1 \cdots h_k(x_k).$$

Because h_k has support of diameter at most k^{-1}, the above inequality implies $d(x_k, x_j) \leq k^{-1}$ if $k \leq j$. Hence $x_k \to x$ for some $x \in X$. But K is a compact set outside which f_0 vanishes. Thus the x_k's and, therefore, x belong to K. Because the h_k's are nonnegative, the last displayed inequality implies $\Sigma_{m=1}^{j} f_m(x_k) < f_0(x_k)$ $(j \leq k)$. Letting $k \to \infty$, we have $\Sigma_{m=1}^{j} f_m(x) \leq f_0(x)$ $(j = 1, 2, \cdots)$. The reasoning up to this point is standard. However, the last inequality is not sufficient to guarantee the constructive convergence of $\Sigma_{m=1}^{\infty} f_m(x)$. Hence the necessity of the following argument.

Let $g \geq 0$ be a function in C which equals 1 on K. Let $a > 0$ be so small that $\Sigma_{m=1}^{\infty} \mu(f_m) + a + a\mu(g) < \mu(f_0)$. Let $N_1 < N_2 < \cdots$ be integers such that $\Sigma_{N_k}^{\infty} \mu(f_m) < 2^{-2k} a$. Then the first part of the proof can be applied to the sequence

$$\left(f_0, ag, f_1, 2\sum_{N_1}^{N_2} f_m, f_2, 2^2 \sum_{N_2}^{N_3} f_m, \cdots \right)$$

to yield $x \in K$ such that

$$ag(x) + f_1(x) + \cdots + f_j(x) + 2^j \sum_{N_j}^{N_{j+1}} f_m(x) \leq f_0(x) \qquad (j = 1, 2, \cdots).$$

In particular

(a) $\Sigma_{N_j}^{N_j+1} f_m(x) \leq 2^{-j} f_0(x)$ and so $\Sigma_1^{\infty} f_m(x)$ converges;

(b) $ag(x) + \Sigma_1^{\infty} f_m(x) \leq f_0(x)$;

(c) $ag(x) = a > 0$ and so $\Sigma_1^{\infty} f_m(x) < f_0(x)$.

The Theorem is proved.

REFERENCE

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