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SOLUTION OF A CONVERGENCE PROBLEM IN THE THEORY OF T-FRACTIONS

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ABSTRACT. Let f be a function, holomorphic in $|z| < R$, where $R > 1$, normalized by $f(0) = 1$, and satisfying a boundedness condition of the form $|f(z) - 1| < K$. It is proved that a certain modification of the Thron continued fraction expansion of f converges to f uniformly on any $|z| \leq r < R$.

Introduction. In 1948 Thron [1] introduced continued fractions of the type

$$1 + d_0 z + \frac{z}{1 + d_1 z} + \cdots + \frac{z}{1 + d_n z} + \cdots. \tag{1}$$

They are called T-fractions. By definition, the T-fraction (1) converges for $z = z_0$ if

$$\lim_{n \to \infty} \left(1 + d_0 z_0 + \frac{z_0}{1 + d_1 z_0} + \cdots + \frac{z_0}{1 + d_n z_0} \right)$$

exists. T-fractions have been studied in [1], [2] and [3]. If we put

$$1 + d_0 z + \frac{z}{1 + d_1 z} + \cdots + \frac{z}{1 + d_n z} = \frac{A_n(z)}{B_n(z)}, \tag{1'}$$

then A_n and B_n can be written as polynomials determined by the recurrence formulas

$$A_{-1}(z) = 1, \qquad A_0(z) = 1 + d_0 z,$$

$$A_n(z) = (1 + d_n z) A_{n-1}(z) + z A_{n-2}(z),$$

$$B_{-1}(z) = 0, \qquad B_0(z) = 1,$$

$$B_n(z) = (1 + d_n z) B_{n-1}(z) + z B_{n-2}(z), \qquad n \geq 1.$$

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Let f_0 be analytic for $z = 0$ and $f_0(0) = 1$. The sequence $\{f_n\}_{n \geq 0}$ is given by

$$(2) \qquad \begin{aligned} &f_n(z) = 1 + (f_n'(0) - 1)z + z/f_{n+1}(z), \qquad z \neq 0, \\ &f_{n+1}(0) = 1, \end{aligned}$$

and f_n is analytic for $z = 0$. If $d_n = f_n'(0) - 1$, the T-fraction (1) is called the T-fraction expansion of f_0.

In [4] H. Waadeland proved: *There exists an* $R_0 > 0$ *such that* $R > R_0$ *implies the existence of a* $K_R > 0$ *with the property*: *If* f_0 *is analytic for* $|z| < R$, $f_0(0) = 1$ *and* $|f_0(z) - 1| < K_R$ *in* $|z| < R$, *then the functions* f_n *given by* (2) *are all analytic in* $|z| < R$. *Furthermore,* $\lim_{n \to \infty} d_n = -1$. It is also shown that $R_0 \geq 1$. The smallest value for R_0 given in [4] is $R_0 = 3/2$.

The quoted result on the sequence $\{f_n\}_{n \geq 0}$ is used in [4] to prove that the corresponding T-fraction expansion of f_0 converges to f_0 locally uniformly in $|z| < 1$. It is pointed out in [5] that this T-fraction expansion never converges to f_0 in a larger disc.

In order to provide convergence in a larger disc a modification of the T-fraction expansion is studied in [5]. This modification is established by replacing the approximants (1′) by modified approximants

$$(3) \quad 1 + d_0 z + \frac{z}{1 + d_1 z} + \cdots + \frac{z}{1 + d_{n-1} z} + \frac{z}{1 + (d_n + 1)z} = \frac{A_n(z) + zA_{n-1}(z)}{B_n(z) + zB_{n-1}(z)}.$$

Under the conditions on f_0 quoted above, the sequence of modified T-approximants converges to f_0 locally uniformly in the disc $|z| < 2R/3$. The question is raised in [5] whether this result can be extended to a result on convergence in the whole disc $|z| < R$. It turns out that the crucial point in the solution of this question is the rate at which f_n tends to 1. The purpose of the present paper is to give an affirmative answer to this question.

A basic convergence property of the sequence $\{f_n\}_{n \geq 0}$. We proceed to prove the following basic result on the sequence $\{f_n\}_{n \geq 0}$, defined by (2).

Theorem 1. *Let* $R > 1$ *and choose* γ *such that* $1 < \gamma < R$. *Then there exists a* $K_R(\gamma) > 0$ *such that if* f_0 *is analytic in* $|z| < R$, $f_0(0) = 1$ *and* $|f_0(z) - 1| < K_R(\gamma)$ *in* $|z| < R$, *there exists a constant* $C_R(\gamma) > 0$ (*independent of* n *and* f_0), *such that*

(4) $$|f_n(z) - 1| \le C_R(\gamma)(\gamma/R)^n$$

for $|z| < R$ *and* $n \ge 0$.

The inequality (4) implies that all f_n are analytic in $|z| < R$ and, by Schwarz' lemma, $\lim_{n\to\infty} d_n = -1$. Besides, $\lim_{n\to\infty} f_n(z) = 1$ uniformly in $|z| < R$. It should be mentioned that (4) fails to hold in the case $R \le 1$ (see [4]). The following proof originated from an analysis of [4, pp. 13–15].

Proof of (4). Let $f_0(z) = 1 + \sum_{q=1}^{\infty} a_q z^q$ for $|z| < R$. Using (2) we obtain for $n \ge 0$

$$f_n(z) = \left(1 + \sum_{q=1}^{\infty} a_q(n) z^q\right) \Big/ \left(1 + \sum_{q=1}^{\infty} b_q(n) z^q\right)$$

(for z in a region Ω_n containing $z = 0$), where

(5) $$\begin{aligned} &a_q(n+1) = b_q(n), \qquad a_q(0) = a_q, \qquad b_q(0) = 0, \\ &b_q(n+1) = a_{q+1}(n) - b_{q+1}(n) - b_q(n)(a_1(n) - b_1(n) - 1). \end{aligned}$$

If for $n \ge 0$ and $q \ge 1$, $c_q(n) = a_q(n) - b_q(n)$, the recurrence formulas (5) show that (for $n \ge 1$, $q \ge 1$)

$$c_q(n+1) = -c_{q+1}(n) - c_1(n) \sum_{k=1}^{n} c_q(k).$$

Using this and $c_q(0) = a_q$, $c_q(1) = -a_{q+1}$, we obtain for $q \ge 1$ and $n \ge 0$

$$c_q(n) = (-1)^n a_{n+q} + s_q(n),$$

where $s_q(n)$ is a linear combination of products (with at least 2 factors) of the numbers $a_1, a_2, a_3, \cdots$. Therefore we obtain for $n \ge 0$

(6) $$f_n'(0) = c_1(n) = (-1)^n a_{n+1} + s_1(n).$$

Let $\mathfrak{F}_0(K)$ be the family of functions f_0 such that f_0 is analytic in $|z| < R$, $f_0(0) = 1$ and $|f_0(z) - 1| < K$ in $|z| < R$ $(K > 0)$. Furthermore, let $\mathfrak{F}_n(K)$ be the family of functions f_n determined by (2) from the functions $f_0 \in \mathfrak{F}_0(K)$. We know that for $q \ge 1$

(7) $$|a_q| \le K/R^q.$$

For $n \ge 0$ the existence of $D_n(K) = \sup_{f_n \in \mathfrak{F}_n(K)} |f_n'(0)|$ is assured by (6), and (6) and (7) give

$$\overline{\lim_{K\to 0}} \frac{D_n(K)}{K} \le \frac{1}{R^{n+1}}. \tag{8}$$

From (2) we get

$$f_{n+1}(z) - 1 = -\frac{(f_n(z)-1)/z - f_n'(0)}{1 + (f_n(z)-1)/z - f_n'(0)} \tag{9}$$

(which in general is meromorphic in $|z| < R$).

Let $m \ge 1$ be an integer such that $1 < \sqrt[m]{m+1} < \gamma < R$. Using (9) we see that there exists a constant $H_1 > 0$ such that if $0 < K < H_1$ and $0 \le n \le m$, the functions in $\mathfrak{F}_n(K)$ are analytic in $|z| < R$ and

$$F_n(K) = \sup_{|z|<R;\, f_n \in \mathfrak{F}_n(K)} |f_n(z) - 1| \tag{10}$$

exists. (9) and (8) then give $(0 \le n \le m)$

$$\overline{\lim_{K\to 0}} \frac{F_n(K)}{K} \le \frac{n+1}{R^n}. \tag{11}$$

Thus, if $\theta \in \langle (m+1)/R^m, 1\rangle$ there exists an $H_2 \in \langle 0, H_1\rangle$ such that $0 < K \le H_2$ and $f_m \in \mathfrak{F}_m(K)$ implies

$$|f_m(z) - 1| < \theta K \tag{12}$$

in $|z| < R$. Because of (12) and (2) we have $\mathfrak{F}_{pm+q}(K) \subseteq \mathfrak{F}_q(K)$ for $p \ge 0$, $0 \le q \le m-1$ and $0 < K \le H_2$. Thus the functions in $\mathfrak{F}_n(K)$ are analytic in $|z| < R$ for $n \ge 0$ $(0 < K \le H_2)$.

Further, according to (12), we have $\mathfrak{F}_{pm+q}(K) \subseteq \mathfrak{F}_q(\theta^p K)$ for $0 < K \le H_2$, $p \ge 0$ and $0 \le q \le m-1$. Therefore the inequality

$$F_q(\theta^p K) \ge \sup_{|z|<R;\, f_n \in \mathfrak{F}_n(K)} |f_n(z) - 1| \tag{13}$$

holds for $0 < K \le H_2$, $n = pm + q$, $p \ge 0$ and $0 \le q \le m-1$. Because of (10) and (11) the existence of the positive number

$$M = \max_{0 \le q \le m-1} \sup_{0 < K \le H_2} \frac{F_q(K)}{K}$$

is assured (observe that $F_q(K_1) \le F_q(K_2)$ for $0 < K_1 \le K_2$). Therefore $F_q(K)$

$\leq M \cdot K$ for $0 < K \leq H_2$ and $0 \leq q \leq m-1$. Thus (for $n = pm + q$, $p \geq 0$ and $0 \leq q \leq m-1$)

$$F_q(\theta^p H_2) \leq M \cdot H_2 \cdot \theta^p \leq MH_2/\theta \cdot \theta^{n/m} = C \cdot \theta^{n/m}.$$

(13) now gives for $n = pm + q$, $p \geq 0$ and $0 \leq q \leq m-1$

$$\sup_{|z|<R;\, f_n \in \mathfrak{F}_n(H_2)} |f_n(z) - 1| \leq C \cdot \theta^{n/m}.$$

As a result we see that if f_0 is analytic in $|z| < R$, $f_0(0) = 1$, $|f_0(z) - 1| < H_2$ and $\{f_n\}_{n\geq 0}$ is determined recursively by (2) from f_0, then $|f_n(z) - 1| \leq C \cdot \theta^{n/m}$ in $|z| < R$. Choosing $\theta = (\gamma/R)^m$ we see that (4) is proved (with $C_R(\gamma) = C$ and $K_R(\gamma) = H_2$).

Application to the modified *T*-fraction expansion. Using (4) and following H. Waadeland [5, pp. 6–10], we are able to conclude the result:

Theorem 2. *Let* $R > 1$ *and* $0 < r < R$. *Then there exists a* $K_r > 0$ *such that if* f_0 *is analytic in* $|z| < R$, $f_0(0) = 1$ *and* $|f_0(z) - 1| < K_r$ *in* $|z| < R$, *then the modified T-fraction expansion converges to* f_0 *uniformly in* $|z| \leq$ $\leq r < R$.

Proof. The difference between $f_0(z)$ and the $(n-1)$th modified approximant may be written

$$f_0(z) - \frac{A_{n-1}(z) + zA_{n-2}(z)}{B_{n-1}(z) + zB_{n-2}(z)} = \frac{(-1)^{n-1}z^n(1 - f(z))}{H_n(z)(H_n(z) + (1 - f_n(z))B_{n-1}(z))}$$

where $H_n(z) = \prod_{k=1}^n f_k(z)$. From (4) and (2) we conclude that the infinite product $\prod_{k=1}^\infty f_k(z)$ converges uniformly on $|z| < R$ and $\prod_{k=1}^\infty f_k(z) \neq 0$ on $|z| < R$.

Let $0 < r < R$ and choose γ such that $r < R/\gamma < R$ and $1 < \gamma < R$. Denote $K_r = K_R(\gamma)$ ($K_R(\gamma)$ in Theorem 1). We then have from (4)

$$|z^n(1 - f_n(z))| \leq C_R(\gamma)(r\gamma/R)^n$$

for $|z| \leq r$.

Further we use

$$B_n(z) - 1 = -z(B_{n-1}(z) - 1)$$
$$+ z\sum_{k=1}^n (1 + d_k)(B_{k-1}(z) - 1) - z + z\sum_{k=1}^n (1 + d_k).$$

This gives for $|z| \le r$ (using (4))

$$|B_n(z) - 1| \le r|B_{n-1}(z) - 1| + r + r \sum_{k=1}^{n} |1 + d_k| + r \sum_{k=1}^{n} |1 + d_k| \cdot |B_{k-1}(z) - 1|$$

$$\le r|B_{n-1}(z) - 1| + r + \frac{r\gamma C_R(\gamma)}{R(R-\gamma)} + \frac{rC_R(\gamma)}{R} \sum_{k=1}^{n} \left(\frac{\gamma}{R}\right)^k |B_{k-1}(z) - 1|.$$

Let $\alpha > 1$ be a number such that $r < \alpha < R/\gamma$. Next we choose n and a constant $G \ge 1$ such that $(1 + 2\gamma C_R(\gamma)/R(R - \alpha\gamma))(1/\alpha^n) \le 1/r - 1/\alpha$ and $|B_k(z) - 1| \le G\alpha^k$ for $0 \le k \le n - 1$. We are now able to conclude that

$$|B_n(z) - 1| \le rG\alpha^{n-1} + r + \frac{r\gamma C_R(\gamma)}{R(R-\gamma)} + \frac{r\gamma C_R(\gamma)G}{R(R-\alpha\gamma)}$$

$$\le Gr\left(\alpha^{n-1} + 1 + \frac{2\gamma C_R(\gamma)}{R(R-\alpha\gamma)}\right) \le Gr\left(\alpha^{n-1} + \alpha^n\left(\frac{1}{r} - \frac{1}{\alpha}\right)\right) = G\alpha^n.$$

By induction we can now easily prove that $|B_k(z) - 1| \le G\alpha^k$ for all $k \ge 0$. Thus we have for $|z| \le r$

$$|(1 - f_n(z))B_{n-1}(z)| \le C_R(\gamma)(\gamma/R)^n(1 + G\alpha^{n-1}) \le 2GC_R(\gamma)(\alpha\gamma/R)^n.$$

From the results obtained we are now able to conclude that

$$\lim_{n\to\infty} \frac{A_{n-1}(z) + zA_{n-2}(z)}{B_{n-1}(z) + zB_{n-2}(z)} = f_0(z)$$

uniformly on $|z| \le r$ and the theorem is proved.

We will now apply Theorem 2 to

Theorem 3. *Let $R > 1$. Let $K_R > 0$ be a number such that f_0 analytic in $|z| < R$, $f_0(0) = 1$ and $|f_0(z) - 1| < K_R$ in $|z| < R$ imply that $\lim_{n\to\infty} f_n(z) =$ 1 uniformly in $|z| < R$. Then the modified T-fraction expansion converges locally uniformly to f_0 in $|z| < R$.*

Proof. Let $0 < r < R$. From Theorem 2 we pick out the number K_r. Now we choose m such that $|f_m(z) - 1| < K_r$ for $|z| < R$. From Theorem 2 we know that the modified T-fraction expansion of f_m converges uniformly to f_m on $|z| \leq r$. Because of the recursive properties of $\{f_n\}_{n \geq 0}$ and the sequence of modified T-fractions we obtain our theorem.

It is clear that the existence of a K_R in Theorem 3 is assured by Theorem 1. To present explicitly a K_R which works requires of course some direct calculation. As an example we will give a K_R for the case $R > 2$ based upon the proof of Theorem 3.1 in [4].

Example. Let $R > 2$. Then all numbers in $\langle 0, R/2 - 1\rangle$ can be used as a K_R in Theorem 3.

Proof. If $0 < K_R < R/2 - 1$, then $0 < 2/(R - 2K_R) < 1$, and it follows easily from (2) that for $n \geq 0$

$$|f_n(z) - 1| < (2/(R - 2K_R))^n K_R$$

in $|z| < R$.

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