PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 2, April 1975

A SEQUENCE-TO-FUNCTION ANALOGUE OF THE HAUSDORFF MEANS FOR DOUBLE SEQUENCES: THE $[J, f(x, y)]$ MEANS

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ABSTRACT. In this paper we extend the Jakimovski $[J, f(x)]$ means to double sequences. We call the new means the $[J, f(x, y)]$ means. We characterize such f's that give rise to regular and to totally regular $[J, f(x, y)]$ means. We also give a necessary and sufficient condition for representability of a function $f(x, y)$ as a double Laplace transform with a determining function of bounded variation in two variables.

1. **Introduction.** Let $f(x, y)$ be a real function of two real variables x, y that has partial derivatives of all orders. The $[J, f(x, y)]$ limit of a double sequence $s_{m,n}$ is

$$\lim_{\substack{x\to\infty \\ y\to\infty}} t(x, y),$$

if it exists, where

$$t(x, y) = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{x^m}{m!} \frac{y^n}{n!} \frac{\partial^{m+n} f}{\partial x^m \partial y^n} s_{m,n}, \tag{1.1}$$

provided that the right-hand side of (1.1) is defined for $x \geq 0$ and $y \geq 0$. We shall denote the first quadrant $\{(x, y): x \geq 0, y \geq 0\}$ by Q.

Let $\alpha(x, y)$ be defined and finite in a rectangle $U = [a, b] \times [c, d]$, and let $a = x_0 < x_1 < \cdots < x_m = b$ and $c = y_0 < y_1 < \cdots < y_n = d$. The double increment of α, say $\Delta(\alpha; x_{i+1}, y_{j+1}; x_i, y_j)$, is

$$\Delta(\alpha; x_{i+1}, y_{j+1}; x_i, y_j) = \alpha(x_{i+1}, y_{j+1}) - \alpha(x_{i+1}, y_j) - \alpha(x_i, y_{j+1}) + \alpha(x_i, y_j).$$

The second variation of α on U, say $V_U[\alpha]$, is

Presented to the Society, December 13, 1972; received by the editors December 14, 1972.

AMS (MOS) subject classifications (1970). Primary 40B05; Secondary 40G05, 44A30.

Key words and phrases. Jakimovski's $[J, f(x)]$ means, regular and totally regular $[J, f(x,y)]$ means, Laplace transforms in two variables.

$$\sup\left\{\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta(\alpha;\, x_{i+1},\, y_{j+1};\, x_i,\, y_j)|\right\},$$

where the supremum is taken over all partitions of U. If $V_U[\alpha]$ is finite one says that $\alpha(x, y)$ is of bounded variation on U. The Stieltjes integral of a function of two real variables is defined similar to the Stieltjes integral of a function of a single real variable. We can always normalize $\alpha(x, y)$ by assuming $\alpha(x, c) = 0,\ a \leq x \leq b,\ \alpha(a, y) = 0,\ c \leq v \leq d.$

Integration over the infinite rectangle Q is defined by

$$\int_Q f(x, y)d\alpha(x, y) = \lim_{X\to\infty,\, Y\to\infty}\int_{(0,0)}^{(X,Y)} f(x, y)d\alpha(x, y).$$

In this paper we prove the following characterizations of regular and totally regular $[J, f(x, y)]$ means.

Theorem 1. *The* $[J, f(x, y)]$ *means are regular if and only if there exists a (normalized) function* $\alpha(x, y)$ *of bounded variation on* Q *such that*

$$f(x, y) = \int_Q e^{-xu-yv}d\alpha(u, v), \tag{1.2}$$

with

$$\int_Q d\alpha(u, v) = 1. \tag{1.3}$$

Theorem 2. *The* $[J, f(x, y)]$ *means are totally regular if and only if the function* $\alpha(u, v)$ *of Theorem* 1 *satisfies*

(i) $\Delta(\alpha;\, x_{i+1},\, y_{j+1};\, x_i,\, y_j) \geq 0,$

(ii) $\alpha(x', y) \leq \alpha(x'', y),\ \alpha(x, y') \leq \alpha(x, y'')$ *for all* $(x, y) \in Q$ *with* $0 \leq x' < x'',\ 0 \leq y' < y'' < \infty.$

In § 2 we prove the above theorems. In § 3 we end the paper by some concluding remarks and characterize real functions $f(x, y)$ that are representable as Laplace transforms, i.e. satisfy (1.2) with $\alpha(u, v)$ of bounded variation on Q.

2. **Regularity and total regularity of the** $[J, f(x, y)]$ **means.** G. M. Robison [5] proved that a sequence-to-function transform T defined by

$$t(x, y) = \sum_{m,n=0}^{\infty} a_{m,n}(x)s_{m,n},$$

where the T-limit of a double sequence $\{s_{m,n}\}$ is

$$\lim_{(x,y)\to(u,v)} t(x, y)$$

(with (u, v) finite or infinite), is regular if and only if

(a) $\lim_{(x,y)\to(u,v)} a_{m,n}(x, y) = 0$ for each m and n;

(b) there exists a finite constant A such that

$$\sum_{m,n=0}^{\infty} |a_{m,n}(x, y)| < A \text{ for all } (x, y);$$

(c) $\lim_{(x,y)\to(u,v)} \sum_{m,n=0}^{\infty} a_{m,n}(x, y) = 1$;

(d) $\lim_{(x,y)\to(u,v)} \sum_{m=0}^{\infty} |a_{m,n}(x, y)| = 0$ for all n, and

(e) $\lim_{(x,y)\to(u,v)} \sum_{n=0}^{\infty} |a_{m,n}(x, y)| = 0$ for all m.

For definitions of regular and totally regular transformations on double sequences, see [5, p. 53]. In particular, note that regularity of a transformation is constructed with regard to convergent bounded sequences.

Proof of Theorem 1. Suppose that (1.2) and (1.3) are satisfied. Then by [1, p. 474]

$$\frac{\partial^{m+n} f(x, y)}{\partial x^m \partial y^n} = \int_Q e^{-ux-vy}(-u)^m(-v)^n d\alpha(u, v) \tag{2.1}$$

and conditions (a) through (e) follow by an easy application of the dominated convergence theorem.

Conversely assume that the $[J, f(x, y)]$ means are regular. Let

$$L_{k,l,s,t}[f] = \frac{(-1)^{k+l}}{k!l!} \frac{\partial^{k+l} f(x, y)}{\partial x^k \partial y^l}\bigg|_{(k/s,l/t)} \left(\frac{k}{s}\right)^{k+1} \left(\frac{l}{t}\right)^{l+1}. \tag{2.2}$$

I claim that there exists a constant M such that

$$\int_Q |L_{k,l,s,t}[f]|\, ds\, dt < M \quad \text{for all } k \geq 0 \text{ and all } l \geq 0. \tag{2.3}$$

This may be proved as follows. Condition (b) implies

$$\sum_{m,n=0}^{\infty} \left|\frac{\partial^{m+n}}{\partial x^m \partial y^n} f(x, y)\right| \frac{x^m}{m!} \frac{y^n}{n!} < A \quad \text{for all } (x, y) \in Q.$$

Using Taylor's formula in two variables (e.g. [7, p. 45]) one can easily show that the remainder in the Taylor series of $\partial^{k+l} f(u, v)/\partial u^k \partial v^l$ about any point (a, b) tends to zero, so that

$$\frac{\partial^{k+l} f(u, v)}{\partial u^k \partial v^l} = \sum_{m,n=0}^{\infty} \frac{(u-a)^m}{m!} \frac{(v-b)^n}{n!} \frac{\partial^{m+l+k+n}}{\partial a^{m+k} \partial b^{n+l}} f(a, b),$$

and hence

$$\int_{(k/a,l/b)}^{(\infty,\infty)} |L_{k,l,s,t}[f]| ds\, dt = \int_{(0,0)}^{(a,b)} \left| \frac{\partial^{k+l} f(u, v)}{\partial u^k \partial v^l} \right| \frac{u^{k-1}}{(k-1)!} \frac{v^{l-1}}{(l-1)!} du\, dv$$

$$\leq \sum_{m,n=0}^{\infty} \left| \frac{\partial^{m+l+n+k}}{\partial a^{m+k} \partial b^{n+l}} f(a, b) \right| \int_{(0,0)}^{(a,b)} \frac{u^{k-1}(a-u)^m v^{l-1}(b-v)^n}{(k-1)!m!(l-1)!m!} du\, dv$$

$$\leq \sum_{m,n=0}^{\infty} \frac{a^m}{m!} \frac{b^n}{n!} \left| \frac{\partial^{m+n} f(a, b)}{\partial a^m \partial b^n} \right| < A,$$

which proves (2.3).

The next step is to notice that

$$\lim_{k,l\to\infty} \int_Q e^{-ux-vy} L_{k,l,u,v}[f] du\, dv = f(x, y), \tag{2.4}$$

whose proof is similar to the proof of Theorem 11a on p. 303 of [6]. In fact, (2.2) and $\lim_{(x,y)\to\infty} f(x, y) = 0$ (by (d)) is all that is required.

Now, set

$$\alpha_{k,l}(s, t) = \int_{(0,0)}^{(s,t)} L_{k,l,u,v}[f] du\, dv.$$

The functions $\alpha_{k,l}(s, t)$ are of uniformly bounded variation by (2.3). By Helly's selection principle generalized to functions of bounded variation (Lemma 1 of [2]) we can find a subsequence $\alpha_{k_i,l_j}(s, t)$ which converges pointwise on Q to a function $\alpha(s, t)$ of bounded variation there. Again by Helly's theorem we have

$$\lim_{(i,j)\to\infty} \int_Q e^{-xu-yv} d\alpha_{k_i,l_j}(u, v) = \int_Q e^{-xu-yv} d\alpha(u, v) \qquad (x > 0, y > 0). \tag{2.5}$$

Therefore (2.5) and (2.4) imply (1.2). In fact (2.1) is also valid, for condition (c) implies

$$\sum_{m,n=0}^{\infty} \int_Q e^{-xu-yv} \frac{(ux)^m}{m!} \frac{(vy)^n}{n!} d\alpha(u, v) = 1, \tag{2.6}$$

and interchanging the summation and integration, by the dominated convergence theorem, proves (1.3).

Proof of Theorem 2. It is clear that if (i) and (ii) are satisfied then

$$\frac{(-x)^m}{m!} \frac{(-y)^n}{n!} \int_Q (-u)^m (-v)^n e^{-xu-yv} d\alpha(u, v) > 0$$

for all $m, n \geq 0$ and all $x > 0$, $y > 0$, and hence the total regularity is obvious.

Conversely, let these means be totally regular. The condition

$$(2.7)\quad \lim_{x\to\infty, y\to\infty} \sum_{m,n=0}^{\infty} \left\{ \frac{x^m}{m!}\frac{y^n}{n!}\left|\frac{\partial^{m+n} f(x,y)}{\partial x^m \partial y^n}\right| - \frac{x^m}{m!}\frac{y^n}{n!}\frac{\partial^{m+n} f(x,y)}{\partial x^m \partial y^n}(-1)^{m+n}\right\} = 0$$

is necessary. This follows by a straightforward extension of Theorem 6 of [3] to double sequences. We have seen in the proof of Theorem 1 that

$$\int_{(k/a,l/b)}^{(\infty,\infty)} |d\alpha_{k,l}(u,v)| = \int_{(k/a,l/b)}^{(\infty,\infty)} |L_{k,l,u,v}[f]|\, du\, dv$$

$$\leq \sum_{m,n=0}^{\infty} \frac{a^m}{m!}\frac{b^n}{n!}\left|\frac{\partial^{m+n} f}{\partial a^m \partial b^n}\right|.$$

Therefore

$$\int_{(0,0)}^{(\infty,\infty)} |d\alpha_{k,l}| \leq \overline{\lim_{a\to\infty; b\to\infty}} \sum_{m,n=0}^{\infty} \frac{a^m}{m!}\frac{b^n}{n!}\left|\frac{\partial^{m+n} f}{\partial a^m \partial b^n}\right|.$$

Therefore, by (2.7),

$$\int_{(0,0)}^{(\infty,\infty)} |d\alpha| \leq \overline{\lim_{a\to\infty; b\to\infty}} \sum_{0}^{\infty} \frac{a^m}{m!}\frac{b^n}{n!}\frac{\partial^{m+n} f(a,b)}{\partial a^m \partial b^n}(-1)^{m+n} = \int_{(0,0)}^{(\infty,\infty)} d\alpha;$$

therefore, α, if normalized, satisfies the requirements of the theorem.

3. **Concluding remarks.** Our means, the $[J, f(x,y)]$ means, are the sequence-to-function analogues to the Hausdorff means for double sequences [2] as the $[J, f(x)]$ means of Jakimovski [4] were the sequence-to-function analogues to the ordinary Hausdorff means. As a matter of fact several of the inclusion relations between different $[J, f(x)]$'s, and between $[J, f(x)]$ and other means, of §§5 and 6 of [4] can be extended to inclusion relations between our $[J, f(x,y)]$ and the respective means by using the same argument.

Our $[J, f(x,y)]$ means include several special well-known means for double sequences. In particular the Abel and Borel (exponential) means are indeed special $[J, f(x,y)]$ means.

Finally it might be worth pointing out that in proving Theorem 1, we have actually proved that a function $f(x,y)$ defined on Q has the representation

$$(3.1)\qquad f(x,y) = \int_Q e^{-xu-yv}\, d\alpha(u,v)$$

with $\alpha(u,v)$ of bounded variation on Q if and only if

$$(3.2)\quad \sum_{m,n=0}^{\infty} \frac{x^m}{m!}\frac{y^n}{n!}\left|\frac{\partial^{m+n} f(x,y)}{\partial x^m \partial y^n}\right| \text{ is uniformly bounded for all } (x,y)\in Q,$$

$$\lim_{x\to\infty} f(x, y) = 0 \quad \text{for all } y \geq 0, \tag{3.3}$$

and

$$\lim_{y\to\infty} f(x, y) = 0 \quad \text{for all } x \geq 0. \tag{3.4}$$

We note that (3.2) implies both (3.3) and (3.4). For, (3.2) implies the existence of an M, independent of $(x, y) \in Q$, such that

$$\sum_0^\infty \frac{x^m}{m!} \left| \frac{\partial^m f(x, y)}{\partial x^m} \right| < M \quad \text{and} \quad \sum_0^\infty \frac{y^n}{n!} \left| \frac{\partial^n f(x, y)}{\partial y^n} \right| < M,$$

so that (3.3) as well as (3.4) follow by Theorems 12a and 13 of Chapter 7 of [6]. Therefore we have proved

Theorem 3. *A real function $f(x, y)$ defined on Q has the representation* (3.1) *with $\alpha(u, v)$ of bounded variation on Q if and only if* (3.2) *is satisfied.*

Acknowledgment. I wish to thank Professor Fred Ustina of the University of Alberta for suggesting this paper's topic, for many fruitful discussions and for reading the manuscript. I also thank Professor Dany Leviatan for correcting an error in an earlier version of the present work.

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