A SEQUENCE-TO-FUNCTION ANALOGUE
OF THE HAUSDORFF MEANS FOR DOUBLE SEQUENCES:
THE \([J, f(x, y)]\) MEANS

MOURAD EL-HOUSSIENY ISMAIL

ABSTRACT. In this paper we extend the Jakimovski \([J, f(x)]\) means to double sequences. We call the new means the \([J, f(x, y)]\) means. We characterize such \(f\)'s that give rise to regular and to totally regular \([J, f(x, y)]\) means. We also give a necessary and sufficient condition for representability of a function \(f(x, y)\) as a double Laplace transform with a determining function of bounded variation in two variables.

1. Introduction. Let \(f(x, y)\) be a real function of two real variables \(x, y\) that has partial derivatives of all orders. The \([J, f(x, y)]\) limit of a double sequence \(s_{m,n}\) is

\[
\lim_{x \to \infty, y \to \infty} t(x, y),
\]

if it exists, where

\[
t(x, y) = \sum_{m, n=0}^{\infty} (-1)^{m+n} \frac{x^m y^n}{m! n!} \frac{\partial^{m+n}f}{\partial x^m \partial y^n} s_{m,n},
\]

provided that the right-hand side of (1.1) is defined for \(x \geq 0\) and \(y \geq 0\). We shall denote the first quadrant \(\{(x, y) : x \geq 0, y \geq 0\}\) by \(Q\).

Let \(\alpha(x, y)\) be defined and finite in a rectangle \(U = [a, b] \times [c, d]\), and let \(a = x_0 < x_1 < \cdots < x_m = b\) and \(c = y_0 < y_1 < \cdots < y_n = d\). The double increment of \(\alpha\), say \(\Delta(\alpha; x_{i+1}, y_{j+1}; x_i, y_j)\), is

\[
\Delta(\alpha; x_{i+1}, y_{j+1}; x_i, y_j) = \alpha(x_{i+1}, y_{j+1}) - \alpha(x_{i+1}, y_j) - \alpha(x_i, y_{j+1}) + \alpha(x_i, y_j).
\]

The second variation of \(\alpha\) on \(U\), say \(V_U[\alpha]\), is

\[
\Delta(\alpha; x_{i+1}, y_{j+1}; x_i, y_j) = \alpha(x_{i+1}, y_{j+1}) - \alpha(x_{i+1}, y_j) - \alpha(x_i, y_{j+1}) + \alpha(x_i, y_j).
\]

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where the supremum is taken over all partitions of $U$. If $V_U[\alpha]$ is finite one says that $\alpha(x, y)$ is of bounded variation on $U$. The Stieltjes integral of a function of two real variables is defined similar to the Stieltjes integral of a function of a single real variable. We can always normalize $\alpha(x, y)$ by assuming $\alpha(x, c) = 0$, $a \leq x \leq b$, $\alpha(a, y) = 0$, $c \leq y \leq d$.

Integration over the infinite rectangle $Q$ is defined by

$$\int_Q f(x, y) d\alpha(x, y) = \lim_{X \to \infty, Y \to \infty} \int_{(0, 0)}^{(X, Y)} f(x, y) d\alpha(x, y).$$

In this paper we prove the following characterizations of regular and totally regular $[J, f(x, y)]$ means.

**Theorem 1.** The $[J, f(x, y)]$ means are regular if and only if there exists a (normalized) function $\alpha(x, y)$ of bounded variation on $Q$ such that

$$f(x, y) = \int_Q e^{-xu-uy} d\alpha(u, v),$$

with

$$\int_Q d\alpha(u, v) = 1.$$  

**Theorem 2.** The $[J, f(x, y)]$ means are totally regular if and only if the function $\alpha(u, v)$ of Theorem 1 satisfies

(i) $\Delta(\alpha; x_{i+1}, y_{j+1}; x_i, y_j) \geq 0$,

(ii) $\alpha(x', y) \leq \alpha(x'', y)$, $\alpha(x', y') \leq \alpha(x, y'')$ for all $(x, y) \in Q$ with $0 \leq x' < x'', 0 \leq y' < y'' < \infty$.

In §2 we prove the above theorems. In §3 we end the paper by some concluding remarks and characterize real functions $f(x, y)$ that are representable as Laplace transforms, i.e. satisfy (1.2) with $\alpha(u, v)$ of bounded variation on $Q$.

2. Regularity and total regularity of the $[J, f(x, y)]$ means. G. M. Robison [5] proved that a sequence-to-function transform $T$ defined by

$$t(x, y) = \sum_{m,n=0}^{\infty} a_{m,n}(x)s_{m,n},$$

where the $T$-limit of a double sequence $\{s_{m,n}\}$ is

$$\lim_{(x,y) \to (u,v)} t(x, y)$$

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(with \((u, v)\) finite or infinite), is regular if and only if

(a) \(\lim_{(x, y)\to (u, v)} a_{m, n}(x, y) = 0\) for each \(m\) and \(n\);
(b) there exists a finite constant \(A\) such that
\[
\sum_{m, n=0}^{\infty} |a_{m, n}(x, y)| < A \quad \text{for all} \quad (x, y);
\]
(c) \(\lim_{(x, y)\to (u, v)} \sum_{m, n=0}^{\infty} a_{m, n}(x, y) = 1\);
(d) \(\lim_{(x, y)\to (u, v)} \sum_{m=0}^{\infty} |a_{m, n}(x, y)| = 0\) for all \(n\), and
(e) \(\lim_{(x, y)\to (u, v)} \sum_{n=0}^{\infty} |a_{m, n}(x, y)| = 0\) for all \(m\).

For definitions of regular and totally regular transformations on double sequences, see [5, p. 53]. In particular, note that regularity of a transformation is constructed with regard to convergent bounded sequences.

**Proof of Theorem 1.** Suppose that (1.2) and (1.3) are satisfied. Then by [1, p. 474]
\[
(2.1) \quad \frac{\partial^{m+n} f(x, y)}{\partial x^m \partial y^n} = \int_Q e^{-u_x - v_y}(-u)^m(-v)^n d\alpha(u, v)
\]
and conditions (a) through (e) follow by an easy application of the dominated convergence theorem.

Conversely assume that the \([f, f(x, y)]\) means are regular. Let
\[
(2.2) \quad L_{k, l, s, \beta, \beta}[f] = \frac{(-1)^{k+l}}{k! l!} \frac{\partial^{k+l} f(x, y)}{\partial x^k \partial y^l} \left| \begin{array}{lcl} \frac{k}{s} & \frac{l}{t} \\ x & y \end{array} \right| \frac{(x, y)}{\alpha(x, y)}.
\]
I claim that there exists a constant \(M\) such that
\[
(2.3) \quad \int_Q |L_{k, l, s, \beta, \beta}[f]| ds dt < M \quad \text{for all} \quad k \geq 0 \quad \text{and all} \quad l \geq 0.
\]
This may be proved as follows. Condition (b) implies
\[
\sum_{m, n=0}^{\infty} \left| \frac{\partial^{m+n} f(x, y)}{\partial x^m \partial y^n} \right| \frac{x^m y^n}{m! n!} < A \quad \text{for all} \quad (x, y) \in Q.
\]
Using Taylor's formula in two variables (e.g. [7, p. 45]) one can easily show that the remainder in the Taylor series of \(\frac{\partial^{k+l} f(u, v)}{\partial u^k \partial v^l}\) about any point \((a, b)\) tends to zero, so that
\[
\frac{\partial^{k+l} f(u, v)}{\partial u^k \partial v^l} = \sum_{m, n=0}^{\infty} \frac{(u-a)^m (v-b)^n}{m! n!} \frac{\partial^{m+l+k+n}}{\partial a^{m+k} \partial b^{n+l}} f(a, b),
\]
and hence
\[
\int_{(k/a, l/b)}^{(\infty, \infty)} |L_{k,l,s,t}[f]| ds \, dt = \int_{(0,0)}^{(a,b)} \left| \frac{\partial^{k+l}f(u, v)}{\partial u^k \partial v^l} \right| \frac{u^{k-1}}{(k-1)!} \frac{v^{l-1}}{(l-1)!} \, du \, dv
\]
\[
\leq \sum_{m,n=0}^{\infty} \left| \frac{\partial^{m+l+n+k}}{\partial a^m b^n} f(a, b) \right| \int_{(a,b)}^{(0,0)} \frac{u^{k-1}(a-u)^m v^{l-1}(b-v)^n}{(k-1)! m!(l-1)!} \, du \, dv
\]
\[
\leq \sum_{m,n=0}^{\infty} \frac{a^m}{m!} \frac{b^n}{n!} \left| \frac{\partial^{m+n}f(a, b)}{\partial a^m \partial b^n} \right| < A,
\]
which proves (2.3).

The next step is to notice that

\[ (2.4) \quad \lim_{k, l \to \infty} \int_{Q} e^{-xu-uy} L_{k,l,u,v}[f] \, du \, dv = f(x, y), \]

whose proof is similar to the proof of Theorem 11a on p. 303 of [6]. In fact, (2.2) and \( \lim_{(x,y) \to \infty} f(x, y) = 0 \) (by (d)) is all that is required.

Now, set

\[ \alpha_{k,l}(s, t) = \int_{(0,0)}^{(s,t)} L_{k,l,u,v}[f] \, du \, dv. \]

The functions \( \alpha_{k,l}(s, t) \) are of uniformly bounded variation by (2.3). By Helly's selection principle generalized to functions of bounded variation (Lemma 1 of [2]) we can find a subsequence \( \alpha_{k,l,i}(s, t) \) which converges pointwise on \( Q \) to a function \( \alpha(s, t) \) of bounded variation there. Again by Helly's theorem we have

\[ (2.5) \quad \lim_{(i,j) \to \infty} \int_{Q} e^{-xu-uy} d\alpha_{k,i,l,j}(u, v) = \int_{Q} e^{-xu-uy} d\alpha(u, v) \quad (x > 0, y > 0). \]

Therefore (2.5) and (2.4) imply (1.2). In fact (2.1) is also valid, for condition (c) implies

\[ (2.6) \quad \sum_{m,n=0}^{\infty} \int_{Q} e^{-xu-uy} \frac{(ux)^m (vy)^n}{m! \, n!} \, d\alpha(u, v) = 1, \]

and interchanging the summation and integration, by the dominated convergence theorem, proves (1.3).

**Proof of Theorem 2.** It is clear that if (i) and (ii) are satisfied then

\[ (-x)^m (-y)^n \int_{Q} (-u)^m (-v)^n e^{-xu-uy} d\alpha(u, v) > 0 \]

for all \( m, n \geq 0 \) and all \( x > 0, y > 0 \), and hence the total regularity is obvious.

Conversely, let these means be totally regular. The condition
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\[
(2.7) \quad \lim_{x \to \infty, \ y \to \infty} \sum_{m,n=0}^{\infty} \left\{ \frac{x^m \ y^n}{m! \ n!} \left[ \frac{\partial^{m+n}f(x, y)}{\partial x^m \partial y^n} \right] - \frac{x^m \ y^n}{m! \ n!} \left[ \frac{\partial^{m+n}f(x, y)}{\partial x^m \partial y^n} \right] (-1)^{m+n} \right\} = 0
\]

is necessary. This follows by a straightforward extension of Theorem 6 of [3] to double sequences. We have seen in the proof of Theorem 1 that

\[
\int_{(0,0)}^{(\infty,\infty)} |d\alpha_k,l(u,v)| = \int_{(k/a,l/b)}^{(\infty,\infty)} |L_k,l,u,v[f]| \, du \, dv
\]

\[
\leq \sum_{m,n=0}^{\infty} \frac{a^m \ b^n}{m! \ n!} \left| \frac{\partial^{m+n}f}{\partial a^m \partial b^n} \right|.
\]

Therefore

\[
\int_{(0,0)}^{(\infty,\infty)} |d\alpha_k,l| \leq \lim_{a \to \infty ; b \to \infty} \sum_{m,n=0}^{\infty} \frac{a^m \ b^n}{m! \ n!} \left| \frac{\partial^{m+n}f}{\partial a^m \partial b^n} \right|.
\]

Therefore, by (2.7),

\[
\int_{(0,0)}^{(\infty,\infty)} |d\alpha| \leq \lim_{a \to \infty ; b \to \infty} \sum_{m,n=0}^{\infty} \frac{a^m \ b^n}{m! \ n!} \left| \frac{\partial^{m+n}f(a,b)}{\partial a^m \partial b^n} \right| (-1)^{m+n} = \int_{(0,0)}^{(\infty,\infty)} d\alpha;
\]

therefore, \( \alpha \), if normalized, satisfies the requirements of the theorem.

3. Concluding remarks. Our means, the \( [J, f(x, y)] \) means, are the sequence-to-function analogues to the Hausdorff means for double sequences [2] as the \( [J, f(x)] \) means of Jakimovski [4] were the sequence-to-function analogues to the ordinary Hausdorff means. As a matter of fact several of the inclusion relations between different \( [J, f(x)] \)'s, and between \( [J, f(x)] \) and other means, of §§5 and 6 of [4] can be extended to inclusion relations between our \( [J, f(x, y)] \) and the respective means by using the same argument.

Our \( [J, f(x, y)] \) means include several special well-known means for double sequences. In particular the Abel and Borel (exponential) means are indeed special \( [J, f(x, y)] \) means.

Finally it might be worth pointing out that in proving Theorem 1, we have actually proved that a function \( f(x, y) \) defined on \( Q \) has the representation

\[
(3.1) \quad f(x, y) = \int_Q e^{-xu-yv}d\alpha(u,v)
\]

with \( \alpha(u, v) \) of bounded variation on \( Q \) if and only if

\[
(3.2) \quad \sum_{m,n=0}^{\infty} \frac{x^m \ y^n}{m! \ n!} \left| \frac{\partial^{m+n}f(x, y)}{\partial x^m \partial y^n} \right| \text{ is uniformly bounded for all } (x, y) \in Q,
\]
(3.3) \[
\lim_{x \to \infty} f(x, y) = 0 \quad \text{for all } y \geq 0,
\]
and

(3.4) \[
\lim_{y \to \infty} f(x, y) = 0 \quad \text{for all } x \geq 0.
\]

We note that (3.2) implies both (3.3) and (3.4). For, (3.2) implies the existence of an \( M \), independent of \((x, y) \in Q\), such that

\[
\sum_{m=0}^{\infty} \frac{x^m}{m!} \left| \frac{\partial^m f(x, y)}{\partial x^m} \right| < M \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{y^n}{n!} \left| \frac{\partial^n f(x, y)}{\partial y^n} \right| < M,
\]

so that (3.3) as well as (3.4) follow by Theorems 12a and 13 of Chapter 7 of [6]. Therefore we have proved

**Theorem 3.** A real function \( f(x, y) \) defined on \( Q \) has the representation (3.1) with \( \alpha(u, v) \) of bounded variation on \( Q \) if and only if (3.2) is satisfied.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA

*Current address:* Mathematics Research Center, University of Wisconsin, Madison, Wisconsin 53706