A SEQUENCE-TO-FUNCTION ANALOGUE
OF THE HAUSDORFF MEANS FOR DOUBLE SEQUENCES:
THE $[J, f(x, y)]$ MEANS
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ABSTRACT. In this paper we extend the Jakimovski $[J, f(x)]$ means to double sequences. We call the new means the $[J, f(x, y)]$ means. We characterize such $f$'s that give rise to regular and to totally regular $[J, f(x, y)]$ means. We also give a necessary and sufficient condition for representability of a function $f(x, y)$ as a double Laplace transform with a determining function of bounded variation in two variables.

1. Introduction. Let $f(x, y)$ be a real function of two real variables $x, y$ that has partial derivatives of all orders. The $[J, f(x, y)]$ limit of a double sequence $s_{m,n}$ is

$$\lim_{x \to \infty} \lim_{y \to \infty} t(x, y),$$

if it exists, where

$$t(x, y) = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{x^m y^n}{m! n!} \frac{\partial^{m+n} f}{\partial x^m \partial y^n} s_{m,n},$$

provided that the right-hand side of (1.1) is defined for $x \geq 0$ and $y \geq 0$. We shall denote the first quadrant $\{(x, y) : x \geq 0, y \geq 0\}$ by $Q$.

Let $\alpha(x, y)$ be defined and finite in a rectangle $U = [a, b] \times [c, d]$, and let $a = x_0 < x_1 < \cdots < x_m = b$ and $c = y_0 < y_1 < \cdots < y_n = d$. The double increment of $\alpha$, say $\Delta(\alpha; x_{i+1}, y_{j+1}; x_i, y_j)$, is

$$\Delta(\alpha; x_{i+1}, y_{j+1}; x_i, y_j) = \alpha(x_{i+1}, y_{j+1}) - \alpha(x_{i+1}, y_j) - \alpha(x_i, y_{j+1}) + \alpha(x_i, y_j).$$

The second variation of $\alpha$ on $U$, say $V_U[\alpha]$, is

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where the supremum is taken over all partitions of $U$. If $V_0[\alpha]$ is finite one says that $\alpha(x, y)$ is of bounded variation on $U$. The Stieltjes integral of a function of two real variables is defined similar to the Stieltjes integral of a function of a single real variable. We can always normalize $\alpha(x, y)$ by assuming $\alpha(x, c) = 0$, $a \leq x \leq b$, $\alpha(a, y) = 0$, $c \leq y \leq d$.

Integration over the infinite rectangle $Q$ is defined by

$$
\int_Q f(x, y) d\alpha(x, y) = \lim_{X \to \infty, Y \to \infty} \int_{(0,0)}^{(X,Y)} f(x, y) d\alpha(x, y).
$$

In this paper we prove the following characterizations of regular and totally regular $[J, f(x, y)]$ means.

**Theorem 1.** The $[J, f(x, y)]$ means are regular if and only if there exists a (normalized) function $\alpha(x, y)$ of bounded variation on $Q$ such that

$$
f(x, y) = \int_Q e^{-xu-yv} d\alpha(u, v),
$$

with

$$
\int_Q d\alpha(u, v) = 1.
$$

**Theorem 2.** The $[J, f(x, y)]$ means are totally regular if and only if the function $\alpha(u, v)$ of Theorem 1 satisfies

(i) $\Delta(\alpha; x_{i+1}, y_{j+1}; x_i, y_j) \geq 0$,

(ii) $\alpha(x', y) \leq \alpha(x'', y)$, $\alpha(x, y') \leq \alpha(x, y'')$ for all $(x, y) \in Q$ with $0 \leq x' < x'', 0 \leq y' < y'' < \infty$.

In §2 we prove the above theorems. In §3 we end the paper by some concluding remarks and characterize real functions $f(x, y)$ that are representable as Laplace transforms, i.e. satisfy (1.2) with $\alpha(u, v)$ of bounded variation on $Q$.
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(with \( (u, v) \) finite or infinite), is regular if and only if

(a) \( \lim_{(x,y) \to (u,v)} a_{m,n}(x, y) = 0 \) for each \( m \) and \( n \);

(b) there exists a finite constant \( A \) such that

\[
\sum_{m, n=0}^{\infty} |a_{m,n}(x, y)| < A \quad \text{for all } (x, y);
\]

(c) \( \lim_{(x,y) \to (u,v)} \sum_{m, n=0}^{\infty} a_{m,n}(x, y) = 1 \);

(d) \( \lim_{(x,y) \to (u,v)} \sum_{m=0}^{\infty} |a_{m,n}(x, y)| = 0 \) for all \( n \), and

(e) \( \lim_{(x,y) \to (u,v)} \sum_{n=0}^{\infty} |a_{m,n}(x, y)| = 0 \) for all \( m \).

For definitions of regular and totally regular transformations on double sequences, see [5, p. 53]. In particular, note that regularity of a transformation is constructed with regard to convergent bounded sequences.

Proof of Theorem 1. Suppose that (1.2) and (1.3) are satisfied. Then by [1, p. 474]

\[
(2.1) \quad \frac{\partial^{m+n}f(x, y)}{\partial x^m \partial y^n} = \int_Q e^{-u x - v y} (-u)^m (-v)^n d\alpha(u, v)
\]

and conditions (a) through (e) follow by an easy application of the dominated convergence theorem.

Conversely assume that the \([f, f(x, y)]\) means are regular. Let

\[
(2.2) \quad L_{k,l,s,t}[f] = \left( \frac{(-1)^{k+l}}{k! l!} \frac{\partial^{k+l} f(x, y)}{\partial x^k \partial y^l} \right) \left( \frac{k}{s} \right) \left( \frac{l}{t} \right)^{l+1}.
\]

I claim that there exists a constant \( M \) such that

\[
(2.3) \quad \int_Q |L_{k,l,s,t}[f]| ds dt < M \quad \text{for all } k \geq 0 \text{ and all } l \geq 0.
\]

This may be proved as follows. Condition (b) implies

\[
\sum_{m, n=0}^{\infty} \left. \frac{\partial^{m+n} f(x, y)}{\partial x^m \partial y^n} \right|_{(x, y) = (u, v)} x^m y^n < A \quad \text{for all } (x, y) \in Q.
\]

Using Taylor's formula in two variables (e.g. [7, p. 45]) one can easily show that the remainder in the Taylor series of \( \frac{\partial^{k+l} f(u, v)}{\partial u^k \partial v^l} \) about any point \((a, b)\) tends to zero, so that

\[
\frac{\partial^{k+l} f(u, v)}{\partial u^k \partial v^l} = \sum_{m, n=0}^{\infty} \frac{(u - a)^m (v - b)^n}{m! n!} \frac{\partial^{m+l+k+n} f(a, b)}{\partial a^{m+k} \partial b^{n+l}}
\]

and hence

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\[
\int_{(0,0)}^{(\infty,\infty)} |L_{k,l,s,t}[f]| ds dt = \int_{(a,b)} \left| \frac{\partial^{k+l} f(u, v)}{\partial u^k \partial v^l} \right| \frac{u^{k-1} v^{l-1}}{(k-1)! (l-1)!} du dv
\]

\[
\leq \sum_{m,n=0}^{\infty} \left| \frac{\partial^{m+n+k} f(a, b)}{\partial a^m \partial b^n} \right| \frac{1}{(a-b)^{m+n+1}} \int_{(0,0)} \frac{u^{k-1} v^{l-1}}{(k-1)! m!(l-1)!} du dv
\]

\[
\leq \sum_{m,n=0}^{\infty} \frac{a^m b^n}{m! n!} \left| \frac{\partial^{m+n} f(a, b)}{\partial a^m \partial b^n} \right| < A,
\]

which proves (2.3).

The next step is to notice that

\[
(2.4) \lim_{k,l \to \infty} \int_Q e^{-u^x - v^y} L_{k,l,u,v}[f] du dv = f(x, y),
\]

whose proof is similar to the proof of Theorem 11a on p. 303 of [6]. In fact, (2.4) and \( \lim_{(x,y) \to \infty} f(x, y) = 0 \) (by (d)) is all that is required.

Now, set

\[
\alpha_{k,l}(s, t) = \int_{(0,0)}^{(s,t)} L_{k,l,u,v}[f] du dv.
\]

The functions \( \alpha_{k,l}(s, t) \) are of uniformly bounded variation by (2.3). By Helly's selection principle generalized to functions of bounded variation (Lemma 1 of [2]) we can find a subsequence \( \alpha_{k,l}(s, t) \) which converges pointwise on \( Q \) to a function \( \alpha(s, t) \) of bounded variation there. Again by Helly's theorem we have

\[
(2.5) \lim_{(i,j) \to \infty} \int_Q e^{-x^u - y^v} d\alpha_{k,l}(u, v) = \int_Q e^{-x^u - y^v} d\alpha(u, v) \quad (x > 0, y > 0).
\]

Therefore (2.5) and (2.4) imply (1.2). In fact (2.1) is also valid, for condition (c) implies

\[
(2.6) \sum_{m,n=0}^{\infty} \int_Q e^{-x^u - y^v} \frac{(ux)^m}{m!} \frac{(vy)^n}{n!} d\alpha(u, v) = 1,
\]

and interchanging the summation and integration, by the dominated convergence theorem, proves (1.3).

Proof of Theorem 2. It is clear that if (i) and (ii) are satisfied then

\[
\frac{(-x)^m}{m!} \frac{(-y)^n}{n!} \int_Q (-u)^m(-v)^n e^{-x^u - y^v} d\alpha(u, v) > 0
\]

for all \( m, n \geq 0 \) and all \( x > 0, y > 0 \), and hence the total regularity is obvious.

Conversely, let these means be totally regular. The condition
is necessary. This follows by a straightforward extension of Theorem 6 of [3] to double sequences. We have seen in the proof of Theorem 1 that

\[
\int_{(k/a, l/b)}^{(\infty, \infty)} |d\alpha_{k,l}(u, v)| = \int_{(k/a, l/b)}^{(\infty, \infty)} |L_{k,l,u,v}[f]| \, du \, dv
\]

\[\leq \sum_{m,n=0}^{\infty} \frac{a^m b^n}{m! n!} \left| \frac{\partial^{m+n} f(a, b)}{\partial a^m \partial b^n} \right|.
\]

Therefore

\[
\int_{(0,0)}^{(\infty, \infty)} |d\alpha_{k,l}| \leq \lim_{a \to \infty; b \to \infty} \sum_{m,n=0}^{\infty} \frac{a^m b^n}{m! n!} \left| \frac{\partial^{m+n} f(a, b)}{\partial a^m \partial b^n} \right|.
\]

Therefore, by (2.7),

\[
\int_{(0,0)}^{(\infty, \infty)} |d\alpha| \leq \lim_{a \to \infty; b \to \infty} \sum_{m,n=0}^{\infty} \frac{a^m b^n}{m! n!} \left| \frac{\partial^{m+n} f(a, b)}{\partial a^m \partial b^n} \right| (-1)^{m+n} = \int_{(0,0)}^{(\infty, \infty)} d\alpha;
\]

therefore, \(\alpha\), if normalized, satisfies the requirements of the theorem.

3. Concluding remarks. Our means, the \([J, f(x, y)]\) means, are the sequence-to-function analogues to the Hausdorff means for double sequences [2] as the \([J, f(x)]\) means of Jakimovski [4] were the sequence-to-function analogues to the ordinary Hausdorff means. As a matter of fact several of the inclusion relations between different \([J, f(x)]\)'s, and between \([J, f(x)]\) and other means, of \(\S \S 5\) and 6 of [4] can be extended to inclusion relations between our \([J, f(x, y)]\) and the respective means by using the same argument.

Our \([J, f(x, y)]\) means include several special well-known means for double sequences. In particular the Abel and Borel (exponential) means are indeed special \([J, f(x, y)]\) means.

Finally it might be worth pointing out that in proving Theorem 1, we have actually proved that a function \(f(x, y)\) defined on \(Q\) has the representation

\[
f(x, y) = \int_{Q} e^{-xu-uy} d\alpha(u, v)
\]

with \(\alpha(u, v)\) of bounded variation on \(Q\) if and only if

\[
\sum_{m,n=0}^{\infty} \frac{x^m y^n}{m! n!} \left| \frac{\partial^{m+n} f(x, y)}{\partial x^m \partial y^n} \right|
\]

is uniformly bounded for all \((x, y) \in Q\),
We note that (3.2) implies both (3.3) and (3.4). For, (3.2) implies the existence of an $M$, independent of $(x, y) \in Q$, such that

$$
\sum_{0}^{\infty} \frac{x^m}{m!} \left| \frac{\partial^m f(x, y)}{\partial x^m} \right| < M \quad \text{and} \quad \sum_{0}^{\infty} \frac{y^n}{n!} \left| \frac{\partial^n f(x, y)}{\partial y^n} \right| < M,
$$

so that (3.3) as well as (3.4) follow by Theorems 12a and 13 of Chapter 7 of [6]. Therefore we have proved

**Theorem 3.** A real function $f(x, y)$ defined on $Q$ has the representation (3.1) with $a(u, v)$ of bounded variation on $Q$ if and only if (3.2) is satisfied.

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