SPHERICAL CURVES AND THEIR ANALOGUES IN AFFINE DIFFERENTIAL GEOMETRY

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ABSTRACT. Necessary and sufficient conditions for curves in Euclidean space to be spherical are derived in a fashion which can be generalized to affine differential geometry and analogues of those curves. This also includes a discussion of some geometrical aspects in recent papers by S. Breuer, D. Gottlieb, and Y.-C. Wong.

1. Introduction. The problem of characterizing classes of ordinary differential equations which can be transformed into equations with constant coefficients was recently considered by S. Breuer and D. Gottlieb [1]. In [2] the authors gave an application to spherical curves by deriving a simple differential equation for the radius of curvature. This result was later used by Y.-C. Wong [9] in connection with his criterion in [8] for a curve to be spherical.

We want to show that the formal approach in [2] has an underlying geometrical idea, which yields the reason for the simplicity of the result in [2], and we shall also give a geometrical motivation and simplified derivation of Wong's result.

Furthermore, that idea can be modified so that it becomes applicable in affine differential geometry. This field was developed by E. Salkowski [6] and others. In a more general and abstract form it has recently become important mainly for functional analytic reasons. In fact, such a "differential geometry of vector spaces" was initiated by E. R. Lorch [4] and R. Nevanlinna [5] and has applications in physics, for instance in the theory of elasticity of anisotropic media. The notion of a sphere does not make sense in that geometry, but we shall define spherical curves in affine space in terms of a property which is characteristic of those curves in Euclidean space. We shall also obtain a condition necessary and sufficient for a curve in affine space to be spherical.
2. Notations. Modified Frenet formulas. Let \( x: J \to E_3 \) represent a curve \( C \) in Euclidean space \( E_3 \), where \( J \subset \mathbb{R} \) is any fixed open interval. We always assume that \( C \) has a unique tangent and positive curvature \( \kappa \) on \( J \) and all appearing derivatives exist and are continuous functions of the arc length \( s \) of \( C \) on \( J \). We also assume that the torsion \( \tau \) of \( C \) is not zero (but shall drop this assumption later). Let \( t, p, b \) be the trihedron of \( C \), and define \( \alpha, \beta, \gamma \) by

\[
\begin{align*}
\alpha &= \kappa ds, \\
\beta &= \sqrt{\kappa^2 + \tau^2} ds, \\
\gamma &= \tau ds;
\end{align*}
\]

these quantities are usually called the angles of contingence of the tangent, principal normal, and binormal, respectively, and \( \gamma \) will play a crucial role in our approach. Setting \( \lambda = \kappa/\tau \), we can easily obtain Frenet formulas for derivatives with respect to \( \gamma \) (denoted by dots):

\[
\begin{align*}
\dot{t} &= \lambda \dot{p}, \\
\dot{p} &= b - \lambda t, \\
\dot{b} &= -p.
\end{align*}
\]

3. Differential equation for \( p \). A curve \( C \) is spherical iff there is a point in common with all normal planes of \( C \). This holds iff there is a cone \( S \) passing through \( C \) and such that any generator of \( S \) through a point \( P \in C \) lies in the normal plane of \( C \) at \( P \). Clearly, \( S \) is a ruled surface which can be represented in the form

\[
\gamma(r, s) = x(s) + r(A(s)p(s) + B(s)b(s)),
\]

where \( x \) represents \( C \). The surface \( S \) is a cone iff there is a function \( r \) of \( s \) such that

\[
x(s) + v(s)p(s) + w(s)b(s) = k,
\]

where \( v(s) = r(s)A(s), w(s) = r(s)B(s), \) and \( k \) is a constant vector. This is equivalent to

\[
\dot{x} + v\dot{p} + v\dot{b} + w\dot{b} = 0.
\]

Applying (2) and \( \dot{x} = t/\tau \), we have

\[
\tau^{-1}t + v\dot{p} + v(b - \lambda t) + \dot{w}b - w\dot{p} = 0.
\]

Equating the coefficients of \( t, \dot{p}, b \) to zero, we have successively

\[
v = \rho, \quad w = \dot{\rho}, \quad v + \dot{w} = 0.
\]

The last relation becomes simply

\[
\ddot{\rho} + \rho = 0.
\]

Solutions are of the form

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where $R$ is the radius of the sphere of $C$. Note that these solutions depend on the torsion. (5) is a necessary and sufficient condition for $C \subset E_3$ to be spherical (even when $\tau = 0$ for some $s$). (4) and (5) were obtained in [2] in an entirely different way, and our derivation shows that the simplicity of the result is achieved because we used the angle of contiguence of the binormal.

4. Another natural equation. A more familiar (but more complicated) natural equation of a spherical curve with $\tau \neq 0$ is readily obtained from (4) and (1):

$$\left(\rho' / r\right)' + \rho \tau = 0,$$

where primes denote derivatives with respect to $s$. Of course other ways of deriving (6) can be modified so that they yield (4). For instance, assuming the formula for the centers of curvature, we have

$$z = x + \rho \hat{p} + (\rho' / r)b = x + \rho \hat{p} + \dot{\rho}b.$$

$C$ is spherical iff $z$ is constant; thus

$$\dot{z} = \dot{x} + \dot{\rho}p + \dot{\rho}p + \dot{\rho}b + \dot{\rho}b = 0.$$

From this and (2), equation (4) follows.

5. On theorems by Y.-C. Wong. A curve $C: x(s)$ with nonzero curvature $\kappa$ and the torsion $\tau$ is spherical iff (6) holds. If $\tau = 0$ at some $s$, then (6) is no longer applicable. This case was considered by Y.-C. Wong [8], [9] who proved the following two theorems.

I. A curve $C: x \in C^4(f)$, $J = [s_1, s_2]$, in $E_3$ with a unique tangent is spherical iff it satisfies the two conditions:

(i) $\kappa(s) > 0$ in $J$;

(ii) there is a function $f \in C^1(f)$ such that

$$f\tau = \rho', \quad f' = -\rho \tau \quad (s \in J).$$

II. The curve $C$ in Theorem I is spherical iff it satisfies (5).

Note that it is not difficult to see that a spherical curve with a unique tangent has positive curvature and thus a unique trihedron.

Wong proved II by showing that (5) satisfies (i) and (ii) and, conversely, can be obtained from (i) and (ii). In [9] he expresses surprise that this is so. We shall give another proof of I which rests on geometrical arguments and thus
Erwin Kreyszig and Alois Pendl explains the geometrical background of Wong's approach and the connection between I and II.

A sphere in $E^3$ having contact of second order with a curve $C: x(s)$ has center $a = x + \rho p + hb$ where $h$ is arbitrary; cf. [3, p. 54]. If for every $s$ we associate with $C$ such a sphere, these spheres have constant radius $|a - x|$ iff

$$\rho \rho' + bb' = 0.$$  

These spheres have contact of third order iff $\rho \kappa' + h \kappa \tau = 0$; cf. [3, p. 54] which is equivalent to

$$(9a) \rho' = b \tau.$$  

From this and (8) we have

$$(9b) \rho r + b' = 0.$$  

If $\tau \neq 0$, we get $b = \rho' / \tau$, and (9b) yields (6) as a necessary and sufficient condition for $C$ to be spherical. If $\tau = 0$ for some $s$, equations (9a) and (9b) still make sense and are precisely the condition (7) (with $f$ denoted by $h$).

Note that we were dealing with osculating spheres, which have contact of third order with the curve, so that Wong's assumption $x \in C^4(f)$ is natural, albeit not the weakest one. Note further that $\tau = 0$ at an $s_0$ implies $\rho' = 0$ at $s_0$; cf. (9a); that is, if the osculating plane is stationary ($b' = 0$) and $C$ is spherical, then $\kappa$ must be stationary at that point. This is geometrically understandable.

To explain the connection between I and II in geometrical terms, we may set $h = \hat{H}$ in (9a) and integrate. Then we see that $\rho = H + c$. Hence the function $f$ in (7) is geometrically the derivative of the radius of curvature with respect to $y$. In (9b) we then have

$$\tau(\rho + \hat{H}) = \tau(\rho + \hat{\rho}) = 0,$$

which gives a reason for the relation between I and II.

6. Some concepts of affine differential geometry. We want to show that the idea of §3 can be generalized to curves in affine space $A^3$. For this we shall need a few simple concepts and facts as follows. Affine differential geometry investigates invariants with respect to the group of those affine transformations

$$x^*_j = \sum_{k=1}^{3} a_{jk} x^*_k + c_j \quad (j = 1, 2, 3)$$

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which are volume-preserving (det$(a_{jk}) = 1$). Such a transformation is said to be equiaffine, and affine differential geometry is also known as equiaffine differential geometry.

Let $x: J \rightarrow A_3$ represent a curve $C$ in $A_3$, where $J = (u_1, u_2) \subset R$ is any fixed open interval. We assume that $x \in C^4(J)$ and $|\dot{x} \dddot{x} \dddot{x}| \neq 0$ on $J$, where $\dot{x} = dx/du$, etc. Then with $C$ we may associate the invariant parameter

$$
\sigma(u) = \int_u^1 |\dot{x} \dddot{x} \dddot{x}|^{1/6} du^* \quad (\cdot = d/du^*)
$$

which is called the affine arc length of $C$ and yields a representation $x(\sigma)$ of $C$. A trihedron of $C$ consists of the tangent vector $t = x'$, the affine normal vector $p = x''$, and the affine binormal vector $b = x'''$; here $x' = dx/d\sigma$, etc. The vectors $p$ and $b$ span the affine normal plane. The affine Frenet formulas are

$$
t' = p, \quad p' = b, \quad b' = -\kappa t - \tau p.
$$

They involve the invariants

$$
\kappa = |x' x''' x^4|, \quad \tau = -|x'' x''' x^4|,
$$

which are called the affine curvature and torsion of $C$.

7. Spherical curves in affine space $A_3$. The property of spherical curves in $E_3$ stated at the beginning of §3 suggests the following

Definition. A curve $C \subset A_3$ is said to be spherical if all affine normal planes of $C$ pass through a common point in $A_3$.

As in §3 this holds iff there is a cone $S \supset C$ such that any generator of $S$ through a point $P \in C$ lies in the affine normal plane of $C$ at $P$. This holds iff there is a function $r$ of $\sigma$ such that in the representation

$$
y(\sigma, \rho) = x(\sigma) + r(\sigma)p(\sigma) + B(\sigma)b(\sigma)
$$

we have

$$
x(\sigma) + v(\sigma)p(\sigma) + w(\sigma)b(\sigma) = k,
$$

where $v(\sigma) = r(\sigma)A(\sigma)$, $w(\sigma) = r(\sigma)B(\sigma)$, and $k$ is a constant vector. This is equivalent to

$$
t + v'p + vp' + wb' + wb' = 0.
$$

Applying (10) and equating to zero the coefficients of the independent vectors $t$, $p$, $b$, we have the three conditions
\[ \dot{\tau} w = 1, \quad \nu' = \kappa w, \quad \nu + w' = 0. \]

A solution is
\[ (11) \quad \kappa = \tau (\tau' / \dot{\tau}^2)'. \]

Performing the indicated differentiation, we could cast this in the form of a Riccati equation, which also appeared in a paper by L. A. Santalo [7] who obtained it in a different way. However, (11) and its derivation, together with suitable differentiability assumptions, give immediately the following remarkable criterion.

**Theorem.** Let \( C: x(u) \) be a curve in affine space \( \mathbb{A}_3 \) which is of class \( C^6(J) \) on some fixed open interval \( J \) and satisfies \( |\dot{x} \times \ddot{x}| \neq 0 \) and \( \ddot{\tau} \neq 0 \) on \( J \), where \( \ddot{\tau} \) is the affine torsion of \( C \). Then \( C \) is spherical (definition above) iff
\[ (12) \quad \chi'' + \kappa(\alpha) \chi = 0, \]
where \( \alpha \) and \( \kappa \) are the affine arc length and affine curvature of \( C \), and \( \chi = 1 / \ddot{\tau} (\ddot{\tau} \neq 0) \) is the affine radius of torsion of \( C \).

A geometrical discussion of (12) will be presented at some other occasion.

**REFERENCES**


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