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CHARACTERIZATIONS OF H-CLOSED SPACES

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ABSTRACT. This paper gives several characterizations of H-closed spaces.

Introduction. Our primary goal is to give some characterizations of H-closed spaces, including a characterization using nets. These characterizations are obtained mainly through the introduction of a type of convergence for filterbases and nets that we call r-convergence.

We also use the technique of [5] to obtain a characterization of H-closed spaces. Following the notation of [5], let S denote a class of topological spaces containing the class of Hausdorff completely normal and fully normal spaces. This characterization is then stated as follows:

Theorem. *A Hausdorff space Y is H-closed if and only if for every space X in class S, each $f: X \to Y$ with a strongly closed graph is weakly continuous.*

Throughout, $\mathrm{cl}(A)$ will denote the closure of a set A and $\mathrm{Int}(A)$ will denote the interior of a set A.

Preliminary definitions and theorems.

Definition 1. The filterbase $F = \{A_\alpha | \alpha \in \Delta\}$ in the topological space X *r-converges* to $x_0 \in X$ $(F \underset{r}{\to} x_0)$ if for each open set V containing x_0 there exists an $A_\alpha \in F$ such that $A_\alpha \subset \mathrm{cl}(V)$.

Definition 2. The filterbase $F = \{A_\alpha | \alpha \in \Delta\}$ in the topological space X *r-accumulates* to $x_0 \in X$ $(F \underset{r}{\propto} x_0)$ if for each $A_\alpha \in F$ and open V containing x_0, $A_\alpha \cap \mathrm{cl}(V) \neq \emptyset$.

Convergence and accumulation of filterbases in the usual sense, of course, imply r-convergence and r-accumulation, respectively. However, the converses do not hold as the next example shows.

Example 1. Let N be the set of positive integers and R be the reals

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with the cocountable topology, and define a filterbase F in R by $F = \{A_n | A_n = \{n, n+1, n+2, \cdots\}, n \in N\}$. For any point $x_0 \in [1, \infty)$, the open set $U = (R - A_1) \cup \{x_0\}$ contains the point x_0, but there does not exist an $A_n \in F$ such that $A_n \subset U$. Consequently, F does not converge to x_0 in the usual sense. However, $F \underset{r}{\rightarrow} x_0$ since the closure of any open set containing x_0 is R. Similarly, F does not accumulate to x_0 in the usual sense but $F \underset{r}{\propto} x_0$.

There are a number of theorems concerning r-convergence and r-accumulation whose statements parallel those of convergence and accumulation in the usual sense. We give some samples of these but omit the proofs, since they are straightforward.

Theorem 1. *Let F be a filterbase in X. If $F \underset{r}{\rightarrow} x_0 \in X$, then $F \underset{r}{\propto} x_0$.*

Theorem 2. *Let F_1 and F_2 be two filterbases in X and suppose F_2 is subordinate to F_1 ($F_2 \prec F_1$). If $F_2 \underset{r}{\propto} x_0$, then $F_1 \underset{r}{\propto} x_0$.*

Theorem 3. *Let F be a maximal filterbase in X. Then $F \underset{r}{\propto} x_0$ if and only if $F \underset{r}{\rightarrow} x_0$.*

Theorem 4. *A filterbase F in $\Pi_\alpha X_\alpha$ r-converges to the point $\{y_\alpha^0\} \in \Pi_\alpha X_\alpha$ if and only if $P_\alpha(F) \underset{r}{\rightarrow} y_\alpha^0$ for every $\alpha \in A$.*

Theorem 5. *If X is Hausdorff, then each r-convergent open filterbase in X r-converges to exactly one point. If each filterbase in X r-converges to exactly one point, then X is Hausdorff.*

For future reference we remark that in Definitions 1 and 2 we could replace the open sets V with regular-open sets [3, p. 92] and have an equivalent definition. The reason is that for any open set V, $\mathrm{Int}(\mathrm{cl}(V))$ is regular open [3, p. 92] and $\mathrm{Int}(\mathrm{cl}(V)) \subset \mathrm{cl}(V)$.

Next we define the concept of r-convergence and r-accumulation for nets. In so doing, the notation of [3, Definition 1.3, p. 210] is used to write $T_a = \{x \in D | a \prec x\}$ where D is a directed set.

Definition 3. Let D be a directed set and $\mathcal{O}: D \rightarrow X$ a net in X. Then

(a) $\mathcal{O}$ *r-converges* to $x_0 \in X$ ($\mathcal{O} \underset{r}{\rightarrow} x_0$) if for each open V containing x_0, there exists an $a \in D$ such that $\mathcal{O}(T_a) \subset \mathrm{cl}(V)$.

(b) $\mathcal{O}$ *r-accumulates* to $x_0 \in X$ ($\mathcal{O} \underset{r}{\propto} x_0$) if for every open V containing x_0 and every $a \in D$, $\mathcal{O}(T_a) \cap \mathrm{cl}(V) \neq \emptyset$.

Of course, if $\mathcal{O}: D \rightarrow X$ is a net in X, the family $F(\mathcal{O}) = \{\mathcal{O}(T_a) | a \in D\}$ is a filterbase in X and it is routine to verify that:

(a) $F(\mathcal{O}) \underset{r}{\rightarrow} x_0$ if and only if $\mathcal{O} \underset{r}{\rightarrow} x_0$, and

(b) $F(\mathcal{O}) \underset{r}{\propto} x_0$ if and only if $\mathcal{O} \underset{r}{\propto} x_0$.

Conversely, every filterbase F in X determines a net $\mathcal{O}: D \rightarrow X$ such that

(a) $F \underset{r}{\rightarrow} x_0$ if and only if $\mathcal{O} \underset{r}{\rightarrow} x_0$, and

(b) $F \underset{r}{\propto} x_0$ if and only if $\mathcal{O} \underset{r}{\propto} x_0$.

The construction of such a net is the same as that of [3, p. 213].

Definition 4. A function $f: X \rightarrow Y$ is weakly continuous (w.c.) [6, p. 44] at $x \in X$ if for each open V containing $f(x)$, there exists an open U containing x such that $f(U) \subset \mathrm{cl}(V)$.

We remark that in Definition 4 we could replace the open set V with regular-open and have an equivalent definition.

Theorem 6. *The function $f: X \rightarrow Y$ is w.c. at $x \in X$ if and only if for every net $\{x_\alpha\}$ in X such that $\{x_\alpha\} \rightarrow x$, the net $\{f(x_\alpha)\} \underset{r}{\rightarrow} f(x)$.*

Main results. Our main results concern H-closed spaces. For a definition of H-closed spaces we use the following:

Definition 5. A Hausdorff space X is H-closed if for every open cover $\{U_\alpha | \alpha \in \Delta\}$ there exists a finite subfamily $\{U_{\alpha_i} | i = 1, 2, \cdots, n\}$ such that $\bigcup_{i=1}^n \mathrm{cl}(U_{\alpha_i}) = X$.

We note here that the open cover in Definition 5 could be replaced with a cover of regular-open sets and an equivalent definition obtained. There are several characterizations of H-closed spaces in the literature. (See, for example [2, p. 145], [1, p. 97].) We offer the following characterizations.

Theorem 7. *Let X be a Hausdorff space. Then the following are equivalent:*

(a) *X is H-closed.*

(b) *For each family of regular-closed sets $\{F_\alpha | \alpha \in \Delta\}$ such that $\bigcap_\alpha F_\alpha = \emptyset$, there exists a finite subfamily $\{F_{\alpha_i} | i = 1, 2, \cdots, n\}$ such that $\bigcap_{i=1}^n \mathrm{Int}(F_{\alpha_i}) = \emptyset$.*

(c) *Each filterbase $F = \{A_\alpha | \alpha \in \Delta\}$ r-accumulates to some $x_0 \in X$.*

(d) *Each maximal filterbase F r-converges.*

Proof. (a) implies (b). Since regular-closed sets are closed, the proof follows from [2, p. 145].

(b) implies (a). Let $\{U_\alpha | \alpha \in \Delta\}$ be a regular-open cover of X. Then $\bigcup_\alpha U_\alpha = X$ so that $\bigcap_\alpha (X - U_\alpha) = \emptyset$. Since each $X - U_\alpha$ is regular-closed, we use the hypothesis to obtain a finite subfamily $\{X - U_{\alpha_i} | i = 1, 2, \cdots, n\}$

such that $\bigcap_{i=1}^{n} \mathrm{Int}(X - U_{\alpha_i}) = \emptyset$. But for each i,

$$\mathrm{Int}(X - U_{\alpha_i}) = X - \mathrm{cl}(X - (X - U_{\alpha_i})) = X - \mathrm{cl}\, U_{\alpha_i},$$

so that $\bigcap_{i=1}^{n} [X - \mathrm{cl}\, U_{\alpha_i}] = \emptyset$. It follows that $\bigcup_{i=1}^{n} \mathrm{cl}\, U_{\alpha_i} = X$ and, consequently, X is H-closed.

(a) implies (c). Suppose there exists a filterbase $F = \{A_\alpha | \alpha \in \Delta\}$ in X that does not r-accumulate in X. Then for each $x \in X$ there exists an open set $V(x)$ containing x and some $A_{\alpha(x)} \in F$ such that $A_{\alpha(x)} \cap \mathrm{cl}(V(x)) = \emptyset$. Now $\bigcup_x \{V(x) |\ x \in X\}$ is an open cover of X, and since X is H-closed there exists a finite subfamily $\{V(x_i) |\ i = 1, 2, \cdots, n\}$ such that $\bigcup_{i=1}^{n} \mathrm{cl}(V(x_i)) = X$. Since F is a filterbase, there exists an $A_{\alpha_0} \in F$ such that $A_{\alpha_0} \subset \bigcap_{i=1}^{n} A_{\alpha(x_i)}$, and $A_{\alpha_0} \neq \emptyset$ implies that for some $1 \leq j \leq n$, $A_{\alpha_0} \cap \mathrm{cl}(V(x_j)) \neq \emptyset$. Therefore, $A_{\alpha(x_j)} \cap \mathrm{cl}(V(x_j)) \neq \emptyset$, which is a contradiction.

(c) implies (b). Let $\{F_\alpha | \alpha \in \Delta\}$ be a family of nonempty regular-closed sets, and suppose for each finite subcollection $\{F_{\alpha_i} |\ i = 1, 2, \cdots, n\}$, we have $\bigcap_{i=1}^{n} \mathrm{Int}(F_{\alpha_i}) \neq \emptyset$. It is easy to verify that the sets $\mathrm{Int}(F_\alpha)$ together with their finite intersections form a filterbase F in X. Our hypothesis now gives the existence of an $x_0 \in X$ such that $F \underset{r}{\propto} x_0$. Thus, for every open V containing x_0 and every $\alpha \in \Delta$, $\mathrm{Int}(F_\alpha) \cap \mathrm{cl}(V) \neq \emptyset$, which implies $F_\alpha \cap \mathrm{cl}(V) \neq \emptyset$ for every α because $F_\alpha = \mathrm{cl}(\mathrm{Int}(F_\alpha))$. Suppose $\bigcap_\alpha F_\alpha = \emptyset$. Then $x_0 \notin \bigcap_\alpha F_\alpha$, so that for some $\beta \in \Delta$, $x_0 \notin F_\beta$ and, consequently, $x_0 \notin \mathrm{Int}(F_\beta)$. Therefore, $x_0 \in X - F_\beta \subset X - \mathrm{Int}(F_\beta)$, and we note that $X - F_\beta$ is regular-open and $X - \mathrm{Int}(F_\beta)$ is regular-closed. It follows that $\mathrm{Int}(F_\beta) \cap \mathrm{cl}(X - F_\beta) = \emptyset$ and this implies the filterbase F does not r-accumulate to x_0, which is a contradiction. We conclude $\bigcap_\alpha F \neq \emptyset$.

(c) implies (d). Let F be a maximal filterbase in X. Then F r-accumulates to some point by (c) and hence r-converges to that point by Theorem 3.

(d) implies (c). Let F be a filterbase in X. Then there exists a maximal filterbase M in X such that $M \prec F$. Since $M \underset{r}{\rightarrow} x_0$ for some $x_0 \in X$ by (d), $F \underset{r}{\propto} x_0$ by Theorems 3 and 2.

Our discussion in the previous section showed that filterbases and nets are "equivalent" in the sense of r-convergence and r-accumulation. Thus we can now characterize H-closed spaces in terms of nets.

Theorem 8. *In a Hausdorff space X the following are equivalent:*

(a) *X is H-closed.*

(b) *Each net in X has an r-accumulation point.*

(c) *Each universal net r-converges.*

Definition 6. A function $f\colon X \to Y$ has a *strongly-closed graph* if for each $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $(U \times \mathrm{cl}(V)) \cap G(f) = \emptyset$.

Clearly any function with a strongly-closed graph also has a closed graph. The converse is not true, however, as we shall see shortly in Example 3.

Example 2. Let X be the closed unit interval with the usual subspace topology and let Y be the closed unit interval with the topology generated by the subspace topology together with the set $A = \{r | r \text{ is rational and } 1/3 < r < 2/3\}$. The identity function $i\colon X \to Y$ has a closed graph and, moreover, the graph is strongly-closed. Note that i is not continuous, but is w.c. Furthermore, the space Y is H-closed but is not semiregular [2, p. 138].

We remark that a map $f\colon X \to Y$ with a strongly-closed graph need not be w.c. For let X be the space given in Example 2 and let Y be the closed unit interval using the usual open sets together with the set $\{1/2\}$ as a subbasis. The identity map $i\colon X \to Y$ has a strongly-closed graph but i is not w.c. at the point $x = 1/2$. Also we note that the space Y is not H-closed. With this observation we now prove a theorem for H-closed spaces which parallels the well-known result that given a compact space Y, for every topological space X, each $f\colon X \to Y$ with a closed graph is continuous.

Theorem 9. *Let Y be an H-closed space. For every topological space X, each $f\colon X \to Y$ with a strongly-closed graph is w.c.*

Proof. Let $x \in X$ and let V be a regular-open set in Y containing $f(x)$. Consider any $y \in Y - V$. Then $(x, y) \notin G(f)$, so there exist open sets $U_y(x) \subset X$ and $V(y) \subset Y$ containing x and y, respectively, such that $[(U_y(x)) \times \mathrm{cl}(V(y))] \cap G(f) = \emptyset$ because $G(f)$ is strongly-closed. Moreover, since Y is Hausdorff, we can choose $V(y)$ such that $f(x) \notin \mathrm{cl}(V(y))$. Now $\{V(y) | y \in Y - V\}$ is an open cover of the regular-closed set $Y - V$ and, since $Y - V$ is H-closed, there is a finite number of these whose closures also cover $Y - V$. We denote this finite cover by $\mathrm{cl}(V(y_1)), \mathrm{cl}(V(y_2)), \cdots, \mathrm{cl}(V(y_n))$. The open set $W = \bigcap_{i=1}^{n} U_{y_i}(x)$ contains no point $w \in W$ such that $f(w) \in \bigcup_{i=1}^{n} \mathrm{cl}(V(y_i))$ since $(Y - V) \subset \bigcup_{i=1}^{n} \mathrm{cl}(V(y_i))$. Therefore $f(W) \subset V \subset \mathrm{cl}(V)$. It follows that f is w.c. at x.

Note that we have actually shown that the function f is almost-continuous, a property stronger than w.c. [8].

The next example shows the strongly-closed graph condition in Theorem 9 cannot be relaxed to a closed graph condition.

Example 3. Let X be the space X of [7, Example 1] and let $A = \{(0, 0)\} \cup \{(1/n, 0) | n \in N\} \subset R \times R$ with the subspace topology. Evidently A is compact and hence is fully normal. Define $f: A \to X$ by $f(1/n, 0) = (1/n, 0) \in X$ and $f(0, 0) = (1, 1)$. Then f is closed with closed point inverses and A is regular so that f has a closed graph according to Corollary 3.9 of [4]. However, f is not w.c. at $(0, 0) \in A$. The reason is that the singleton $\{(1, 1)\}$ is both open and closed in X, and hence any open $U \subset A$ containing $(0, 0)$ has the property that $f(U) \not\subset \mathrm{cl}\{(1, 1)\} = \{(1, 1)\}$.

We next show that even though the graph of f is closed in $A \times X$, it is not strongly-closed. Now consider the point $((0, 0), (0, 0)) \in A \times X$. Since $G(f)$ is closed in $A \times X$, there exist open sets $U \subset A$ and $V \subset X$ containing $(0, 0) \in A$ and $(0, 0) \in X$, respectively, such that $[U \times V] \cap G(f) = \emptyset$. However, it follows from the construction of the topology on X that $[U \times \mathrm{cl}(V)] \cap G(f) \neq \emptyset$. Hence, $G(f)$ is not strongly-closed in $A \times X$.

Our final and main result shows that the condition of Theorem 9 characterizes H-closed spaces if the spaces X are chosen from a particular class of topological spaces. Professor Kasahara [5] denotes by S a class of spaces containing the class of Hausdorff completely normal and fully normal spaces and uses the class S to obtain a characterization of compact spaces. We use the same class S in our characterization.

Theorem 10. *A Hausdorff space Y is H-closed if and only if for every space $X \in S$, each $f: X \to Y$ with a strongly closed graph is w.c.*

Proof. In view of Theorem 9, only the sufficiency needs proof.

Assume Y is not H-closed. We proceed to construct a space X in the class S and a function $f: X \to Y$ which has a strongly-closed graph but is not w.c. Since Y is not H-closed, there exists a net $f: D \to Y$ which has no r-accumulation point in Y. Thus, for each $y \in Y$ there is an open $V(y)$ and some $a \in D$ such that $f(T_a) \cap \mathrm{cl}(V(y)) = \emptyset$.

Now choose a point $\infty \notin D$ and let $X = D \cup \{\infty\}$. Then the power set of D, $P(D)$, together with $\{T_a \cup \{\infty\} | a \in D\}$ is a base for a topology σ on X. It follows from [5] that (X, σ) is a Hausdorff fully normal space and furthermore has the property that if $U(\infty)$ is an open set containing ∞, then there exists an $a \in D$ such that $T_a \cup \{\infty\} \subset U(\infty)$.

Let $b \in Y$ and define $g: X \to Y$ by $g|D = f$ and $g(\infty) = b$. Let $(x, y) \notin G(g)$ ($G(g)$ is the graph of g) and first consider the case when $x \in D$. Since $g(x) \neq y$ and Y is Hausdorff, there exists a regular-open $V(y)$ containing y such that $g(x) \notin \mathrm{cl}(V(y))$. Now $\{x\}$ is open and closed in (X, σ), which implies $(\{x\} \times V(y))$ is open in $X \times Y$. Furthermore, $(\{x\} \times V(y)) \cap G(g) = \emptyset$ and $(\{x\} \times \mathrm{cl}(V(y))) \cup G(g) = \emptyset$. Now consider the case where $(x, y) \notin G(g)$ and $x = \infty$. Then $g(\infty) = b \neq y$ and there exists a regular-open set $V(y)$ containing y such that $b \notin \mathrm{cl}(V(y))$. Since y is not an r-accumulation point of the net $f: D \longrightarrow Y$, there exists a regular-open set $W(y) \subset V(y)$ and an $a \in D$ such that $\mathrm{cl}(W(y)) \cap f(T_a) = \emptyset$. But $g|D = f$ which implies $\mathrm{cl}(W(y)) \cap g(T_a) = \emptyset$, so that $(T_a \cup \{\infty\} \times \mathrm{cl}(W(y))) \cap G(g) = \emptyset$. We have now proved $G(g)$ is strongly closed.

Now the identity function $i: D \longrightarrow D \subset X$ defines a net in X. From the nature of the topology σ on X, it follows that the net i converges in the usual sense to ∞ in X. That is, given an open set $H(\infty)$ containing ∞, there exists an $a \in D$ such that $i(T_a) = T_a \subset H(\infty)$. Since $b \in Y$ is not an r-accumulation point of the net f, there exist an open set $V(b)$ and some $a_0 \in D$ such that for each $a > a_0$, $f(T_a) \cap \mathrm{cl}(V(b)) = \emptyset$. But by definition of g, $f(T_a) = g(T_a)$ so that $g(T_a) \cap \mathrm{cl}(V(b)) = \emptyset$, which implies g is not w.c. according to Theorem 6.

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