IMBEDDING COMPACT 3-MANIFOLDS IN $E^3$

TOM KNOBLAUCH

ABSTRACT. We show that in a large finite disjoint collection of compacta in a closed orientable 3-manifold there is a compactum that imbeds in $E^3$. However, given a closed 3-manifold $M^3$, there is a pair of compact 3-manifolds $(L, N)$ such that $L$ contains infinitely many disjoint copies of $N$ but $N$ does not imbed in $M^3$.

It is sometimes useful to know that a subset of a 3-manifold can be imbedded in $E^3$. For example, it is sometimes desirable to use a linking argument in a neighborhood of an element of a decomposition. We will prove that in a large enough disjoint collection of compacta in a closed orientable 3-manifold there is a compactum with a neighborhood that imbeds in $E^3$. We will then show that no closed 3-manifold is an imbedding manifold for nonorientable 3-manifolds in the sense that $E^3$ is for orientable 3-manifolds.

Definitions. A surface is a closed connected 2-manifold. A surface $S$ is incompressible in the 3-manifold $M^3$ means (1) $S$ is not a 2-sphere and if $D \subseteq M^3$ is a disk with $D \cap S = \text{Bd} D$, then $\text{Bd} D$ bounds a disk in $S$; or (2) $S$ is a 2-sphere that bounds no 3-cell in $M^3$. The surfaces $S_1$ and $S_2$ are said to bound a parallelity component $U$ of $M^3$ and the disjoint collection of surfaces $\{S_1, S_2, \ldots, S_n\}$ if $U$ is a component of $M^3 - \bigcup_{i=1}^{n} S_i$, $\overline{U}$ is homeomorphic to $S_1 \times I$, and $\text{Bd} \overline{U} = S_1 \cup S_2$.

We assign some constants to a given compact 3-manifold $M^3$. Let $\beta_1(H_1(M^3, G))$ be the rank of $H_1(M^3, G)$. When $G = \mathbb{Z}$ we sometimes write just $\beta_1$. Also let $\alpha$ be the maximum number of disjoint nonparallel incompressible 2-sided surfaces in $M^3$. We know $\alpha$ exists by [1].

Theorem 1. Suppose $M^3$ is a closed orientable 3-manifold and $\{C_1, \ldots, C_{a+\beta_1+2}\}$ is a disjoint collection of compacta in $M^3$. Then there is a $C_i$ with a neighborhood imbeddable in $E^3$.

Proof. Let $\{U_1, \ldots, U_{a+\beta_1+2}\}$ be a disjoint collection of compact poly-
hedral 3-manifolds such that $U_i$ is a neighborhood of $C_i$ for each $i$. We will show that some $U_i$ imbeds in $E^3$.

We first alter the $U_i$'s. If for some $i$ there is a component of $\partial U_i$ that is a compressible surface not a 2-sphere, then we can find a disk $D$ so that $D \cap \partial U_j = \partial D$ for some $j$, and either $D \subseteq U_j$ or $D \cap (\bigcup_{i=1}^{a+b_1+2} U_i) = \partial D$.

In the first case we alter the collection by replacing $U_j$ by $U_j - N(D)$, where $N(D)$ is a regular neighborhood of $D$ in $U_j$, and in the second case we replace $U_j$ by $U_j \cup N(D)$, where $N(D)$ is a regular neighborhood of $D$ in $M^3 - \bigcup_{i=1}^{a+b_1+2} U_i$. Since each such step reduces the sum $\Sigma_{S \in C} (x(S) - 2)^2$, where $C$ the collection of surfaces in $\bigcup_{i=1}^{a+b_1+2} \partial U_i$ (where $x$ stands for Euler characteristic), in a finite number of steps we get a disjoint collection $\{V_1, \ldots, V_{a+b_1+2}\}$ of compact 3-manifolds so that each component of each $\partial V_i$ is incompressible in $M^3$ or a 2-sphere bounding a 3-cell.

**Lemma 2.** If $M^3$ is a closed 3-manifold and $\{V_1, \ldots, V_{a+b_1+2}\}$ is a disjoint collection of compact connected 3-manifolds in $M^3$ such that each component of each $\partial V_i$ is incompressible in $M^3$, then there is an $i_0$ such that $V_{i_0}$ is the product of a surface and an interval.

**Proof.** Let

$$\{S_1, \ldots, S_{a+2+k}\}$$

be the collection of boundary components of the $V_i$'s. We know $M^3 - \bigcup_{i=1}^{a+b_1+2} S_i$ has at most $a + 3 + k$ components, so $M^3 - \bigcup_{i=1}^{a+b_1+2} V_i$ has at most $k + 1$ components $\{W_1, \ldots, W_l\}$ with $l \leq k + 1$. We now change the collection (1) by removing, for each $W_j$, one surface $S_i \subset \partial W_j$. Our collection still contains at least $a + 1$ surfaces and so there are two surfaces, call them $S_1$ and $S_2$, such that $S_1$ and $S_2$ bound $V$, a parallelity component of the changed collection [1]. Notice that $S_1 \cup S_2$ is not the entire boundary of any $W_j$. Then there is an $i_0$ such that $V_{i_0} \subset \overline{V}$. Now by [1, p. 91], $\partial V_{i_0}$ has exactly two components and $V_{i_0}$ is topologically the product of a surface and an interval.

Let $\{X_1, X_2, \ldots, X_{a+b_1+2}\}$ be a collection of components of $\bigcup_{i=1}^{a+b_1+2} V_i$. We claim that some $X_i$ imbeds in $E^3$. We first consider all components of $\bigcup_{i=1}^{a+b_1+2} Bd X_i$. If some component is a 2-sphere bounding a 3-cell, then either that 3-cell contains an $X_i$ or else we can add the 3-cell to the appropriate $X_j$ to get a new collection of $a + 2$ disjoint compact connected 3-manifolds. Eventually we have either some $X_i$ in a 3-cell or all the boundary components incompressible and Lemma 2 applies to establish the claim.
It follows from the claim that at most $\alpha + 1$ of the $V_i$'s have components that do not imbed in $E^3$, and therefore at least $\beta_1 + 1$ of the $V_i$'s obtained from the $U_i$'s can be imbedded in $E^3$. So we may assume that each of

$$\{V_1, \ldots, V_{\beta_1 + 1}\}$$

can be imbedded in $E^3$.

Notice that we can go step by step in $M^3$ from the collection (2) to the collection

$$\{U_1, \ldots, U_{\beta_1 + 1}\}$$

by performing the operations preceding Lemma 2 backwards, so that each step consists of adding a 1-handle to a compact 3-manifold or removing a regular neighborhood of a properly imbedded arc from (digging a tunnel in) a compact 3-manifold.

**Lemma 3.** If we reconstruct (3) from (2) as above, then there is an $i_0$ so that in the reconstruction of $U_{i_0}$, whenever we attach a 1-handle to a connected compact 3-manifold, we attach both ends to the same boundary component.

**Proof.** Suppose that in the reconstruction of $U_i$ for each $i$, we attach a 1-handle $H_i$ to the compact connected 3-manifold $M_i$ with one end on boundary component $S_i$ and one end on another boundary component. Then we can draw a simple closed curve $J_i \subset M_i \cup H_i$ made up of an arc in $H_i$ and an arc in $M_i$ so that the intersection number of $J_i$ and $S_i$ is one. In subsequent steps we dig all tunnels to miss $J_i$.

Since we have more than $\beta_1$ curves, there is a nontrivial relation in $H_1(M^3, Z)$,

$$n_1[J_1] + \cdots + n_{\beta_1 + 1}[J_{\beta_1 + 1}] = 0,$$

where $[J]$ is the element of $H_1(M^3, Z)$ represented by $J$. We may assume $n_1 \neq 0$ and that $J_1$ is the last such curve formed in the sequence going from (2) to (3). The relation tells us that the intersection number of $S_1$ and the 1-manifold $n_1 J_1 \cup \cdots \cup n_{\beta_1 + 1} J_{\beta_1 + 1}$ is zero, where $n_i J_i$ is the 1-manifold consisting of $n_i$ copies of $J_i$ near $J_i$. But $J_i \cap S_1 = \emptyset$ for $i \neq 1$ and the intersection number of $J_1$ and $S_1$ is one. We have arrived at a contradiction and the lemma is proved.

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We now complete the proof of Theorem 1. For each $V_i$ in the collection (2) we attempt to mimic in $E^3$ the step by step reconstruction of $U_i$ from $V_i$ occurring in $M^3$.

There is no problem if the ends of a 1-handle are to be attached to different components of a compact 3-manifold; we imbed one component in $E^3$, and then we imbed the other component in the proper complementary domain of the first, being careful to imbed so that the boundary components to which the 1-handle is to be attached are adjacent.

We face a problem only if the ends of a 1-handle are attached to different boundary components of a compact connected 3-manifold. By Lemma 3 we may assume this problem never occurs in the construction of $U_i$ from $V_i$, and so $U_1$ can be imbedded in $E^3$.

Notice that $U_1$ is constructed from the product of orientable surfaces with an interval by removing 3-cells from the interior, adding 1-handles, and drilling tunnels.

The referee has pointed out the following corollary to Theorem 1.

Corollary. Let $M^3$ be a closed 3-manifold that is not sufficiently large (that is, $M^3$ contains no incompressible surfaces). If $X_1$ and $X_2$ are disjoint compacta in $M$, then one of $X_1$ and $X_2$ imbeds in $E^3$.

Proof. It follows from [2, p. 774] that $M^3$ is orientable and $\beta_1(M^3) = 0$. Since $M^3$ is not sufficiently large, $\alpha = 0$ and the Corollary follows from Theorem 1.

In contrast, if we drop the orientability hypothesis from Theorem 1, then given a closed 3-manifold $W^3$, we can construct a pair of compact 3-manifolds $(L, N)$ with the following properties:

1. $L$ contains infinitely many disjoint copies of $N$, and
2. $N$ cannot be imbedded in $W^3$.

We first construct a sequence of auxiliary pairs $\{(L_i, N_i)\}$.

We let $N_1 = L_1 = P \times I$, the product of a projective plane and an interval. If $i \geq 2$, the construction of $L_i$ begins with $i$ copies of $P \times I$. We label them $\{P_j \times I | 1 \leq j \leq i\}$. We then connect $P_j \times 0$ with $P_1 \times 0$ by a 1-handle $H_j$ for each $j$ with $2 \leq j \leq i$. This is $L_i$. We construct $N_i$ from a copy of $L_i$ by removing regular neighborhoods of $i - 1$ disjoint arcs $\{A_j | 2 \leq j \leq i\}$, where $A_j$ has one end in $P_j \times 1$ and one end in $P_1 \times 1$. The arc $A_j$ runs from $P_j \times 1$ through $H_j$, around an orientation reversing curve in $P_1 \times I$, and back through $H_j$, around an orientation reversing curve in $P_j \times I$, and back through $H_j$ to $P_1 \times I$. 

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Now given $W^3$ let

$$K = \beta_1(W^3, Z_3).$$

Then we choose $N$ to be the disjoint union

$$N = N_{2K+1} \cup N_{4K+2} \cup \cdots \cup N_{2K+1}K + 1K + 2K,$$

and we let $L$ be the disjoint union

$$L = L_{2K+1} \cup L_{4K+2} \cup \cdots \cup L_{2K+1}K + 2K.$$

\textbf{Property (1).} To prove that $(L, N)$ has Property 1 it is clearly enough to imbed infinitely many disjoint copies of $N_i$ in $L_i$ for a given $i$. As above we write $L_i$ as

$$L_i = \left( \bigcup_{j=1}^{i} P_j \times l_{j} \right) \cup \left( \bigcup_{j=1}^{i} H_j \right).$$

We construct $N_{i,1}$, our first copy of $N_i$, starting with $P_{1} \times [\frac{1}{2}, \frac{1}{2}]$. Then $P_{1} \times \frac{1}{2}$ is joined to $P_{j} \times \frac{1}{2}$ by a 1-handle running through $H_j$ for $2 \leq j \leq i$. The tunnels are drilled by removing appropriate arcs. The $N_{i,2}$ starts with $\bigcup_{j=1}^{i} P_j \times [\frac{3}{4}, \frac{7}{8}]$. Then $P_{1} \times \frac{3}{4}$ is joined to $P_j \times \frac{3}{4}$ by a 1-handle running through the $j$th tunnel in $N_{i,1}$ for each $j$ with $2 \leq j \leq i$. The tunnels are then drilled to complete $N_{i,2}$. The rest of the construction is clear.

\textbf{Property (2).} Suppose $N$ can be imbedded in $W^3$. Let $B_{1,2i+2i-1}$ be the boundary component of $N_{2i+2i-1}$ that intersects $P_{1} \times 1$ for each $i$ with $i = 1, 2, \cdots, K + 1$, and let $B_{2,2i+2i-1}$ be the other boundary component of $N_{2i+2i-1}$. By our choice of $K$ there is a smallest $r$ such that $B_{1,2r+2r-1}$ separates $W^3 \setminus \bigcup_{i=1}^{r-1} B_{1,2i+2i-1}$. We let

$$B_1 = B_{1,2r+2r-1}, \quad B_2 = B_{2,2r+2r-1} \quad \text{and} \quad N' = N_{2r+2r-1}.$$

Then we let $U$ and $V$ be the components of $M^3 \setminus \bigcup_{i=1}^{r-1} B_{1,2i+2i-1}$ with $N' \subseteq \overline{U}$.

\textbf{Lemma 4.} $\beta_1(H_1(\overline{U}, Z_3)) \leq \beta_1(H_1(\overline{U} \cap \overline{V}, Z_3)) + \beta_1(H_1(\overline{U} \cup \overline{V}, Z_3)) - \beta_1(H_1(\overline{V}, Z_3)).$

\textbf{Proof.} The Mayer-Vietoris sequence

$$\to H_1(\overline{U} \cap \overline{V}, Z_3) \xrightarrow{i^*} H_1(\overline{U}, Z_3) \oplus H_1(\overline{V}, Z_3) \xrightarrow{j^*} H_1(\overline{U} \cup \overline{V}, Z_3) \to$$
generates the short exact sequence

\[ 0 \to H_1(\tilde{U} \cap \tilde{V}, \mathbb{Z}_3)|\text{Ker } i^* \to H_1(\tilde{U}, \mathbb{Z}_3) \oplus H_1(\tilde{V}, \mathbb{Z}_3) \to \text{Im } j^* \to 0, \]

so we can think of \( H_1(\tilde{U} \cap \tilde{V}, \mathbb{Z}_3)|\text{Ker } i^* \) as a subgroup of \( H_1(\tilde{U}, \mathbb{Z}_3) \oplus H_1(\tilde{V}, \mathbb{Z}_3) \) and

\[ (H_1(\tilde{U}, \mathbb{Z}_3) \oplus H_1(\tilde{V}, \mathbb{Z}_3))|(H_1(\tilde{U} \cap \tilde{V}, \mathbb{Z}_3)|\text{Ker } i^*) = \text{Im } j^*. \]

Then \( H_1(\tilde{U}, \mathbb{Z}_3) \oplus H_1(\tilde{V}, \mathbb{Z}_3) \) has as many generators as \( H_1(\tilde{U} \cap \tilde{V}, \mathbb{Z}_3)|\text{Ker } i^* \) and \( \text{Im } j^* \) combined, which implies the conclusion of the lemma.

**Lemma 5.** If \( \tilde{V} \) is a compact 3-manifold with nonempty boundary, then

\[ \beta_1(H_1(\tilde{V}, \mathbb{Z}_3)) \geq 1 - \frac{1}{2} \times (\text{Bd } \tilde{V}). \]

**Proof.** We know by [3, p. 233] that \( 2\chi(\tilde{V}) = \chi(\text{Bd } \tilde{V}) \) (where \( \chi \) stands for Euler characteristic). For compact 3-manifolds, \( \chi(\tilde{V}) = \beta_0 - \beta_1 + \beta_2 - \beta_3 \) and \( \beta_0 \geq 1 \) and \( \beta_3 = 0 \). So we have \( 1 - \beta_1 + \beta_2 \leq \frac{1}{2}\chi(\text{Bd } \tilde{V}) \) or \( \beta_1 \geq 1 - \frac{1}{2}\chi(\text{Bd } \tilde{V}) \).

By Lemmas 4 and 5 we have

\[ \beta_1(H_1(\tilde{U}, \mathbb{Z}_3)) \leq \sum_s (2^iK + 2^{i-1} - 1) + K - \left( 1 + \frac{1}{2} \sum_s (2^iK + 2^{i-1} - 2) \right) \]

(\text{where } i \in S \text{ if } B_{1, 2^iK + 2^{i-1} - 1} \subseteq \text{Bd } \tilde{V})

\[ = \frac{1}{2} \left( \sum_s (2^iK + 2^{i-1} - 2) \right) + K + |S| - 1 \]

\[ \leq \frac{1}{2} \left( \sum_{i=1}^r 2^iK + 2^{i-1} - 2 \right) + K + r - 1 = 2^rK + 2^{r-1} - 1\frac{1}{2}. \]

We know that \( \beta_1(H_1(B_1, \mathbb{Z}_3)) = 2^rK + 2^{r-1} - 1 \), so there is a \( g \in H_1(B_1, \mathbb{Z}_3) \) with \( g \neq 0 \) and \( j_\#g = 0 \) in \( H_1(\tilde{U}, \mathbb{Z}_3) \), where \( j_\# \) is the inclusion induced homomorphism.

Now consider the Mayer-Vietoris sequence

\[ \to H_1(B_2, \mathbb{Z}_3) \xrightarrow{\tau_*} H_1(N', \mathbb{Z}_3) \oplus H_1(\tilde{U} \cap N', \mathbb{Z}_3) \xrightarrow{s_*} H_1(\tilde{U}, \mathbb{Z}_3). \]

Since \( s_*(i_\#g, 0) = j_\#g = 0 \), where \( i_* \) is here the homomorphism induced by the inclusion of \( \text{Bd } N' \) in \( N' \), there is an \( l \in H_1(B_2, \mathbb{Z}_3) \) such that \( \tau_\#l = (i_\#g, 0) \) or \( i_*l = i_\#g \) in \( H_1(N', \mathbb{Z}_3) \).

We now investigate \( H_1(N', \mathbb{Z}_3) \). Let \( a_j \) be the orientation reversing
curve in $P_j \times 1$ and $b_j$ be the orientation reversing curve in $P_j \times 0$ for each $j$ with $1 \leq j \leq 2^rK + 2^{r-1}$. If the curves are oriented properly and if $[a_j]$ and $[b_j]$ represent the homology classes of $a_j$ and $b_j$ in $H_1(N', \mathbb{Z})$, then $[b_j] = 3[a_j]$ in $H_1(N', \mathbb{Z})$. Therefore $[b_j] = 0$ in $H_1(N', \mathbb{Z}_3)$. Since the $[b_j]$'s generate $H_1(B_2, \mathbb{Z}_3)$ we have $i_*H_1(B_2, \mathbb{Z}_3) = 0$ in $H_1(N', \mathbb{Z}_3)$ and $i_*g = i_*l = 0$ in $H_1(N', \mathbb{Z}_3)$ so $\beta_1(H_1(N', \mathbb{Z}_3)) \leq 2^rK + 2^{r-1} - 1$. But since $i_*g = 0$ in $H_1(N', \mathbb{Z}_3)$,

$$\beta_1(H_1(N', \mathbb{Z}_3)) \leq 2^rK + 2^{r-1} - 2.$$ 

However by Lemma 5

$$\beta_1(H_1(N', \mathbb{Z}_3)) \geq \beta_1(H_1(N', \mathbb{Z})) \geq 1 - \frac{1}{2}\chi(Bd N') = 1 - \frac{1}{2}(2(2 - 2^rK - 2^{r-1})) = 2^rK + 2^{r-1} - 1.$$ 

The last two inequalities are incompatible, and Property (2) is established.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, BINGHAMTON, NEW YORK 13901

Current address: Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73069