PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 2, April 1975

A NOTE ON THE RELATIONSHIP BETWEEN WEIL AND CARTIER DIVISORS

JAMES HORNELL

ABSTRACT. Using a generalized equivalence relation, a subquotient of the group of Weil divisors is shown to be isomorphic to the group of Cartier divisors modulo linear equivalence for a reduced subscheme of a projective space over a field. A difficulty of the nonreduced case is discussed.

Let X, $\mathcal{O}$ be a subscheme of a projective space over a field. A generalized equivalence relation is defined on the group of Weil divisors, and if X, $\mathcal{O}$ is reduced, a corresponding subquotient is shown to be isomorphic to the group of Cartier divisors modulo linear equivalence. The generalized equivalence for curves appears in [5]. Projectivity may be replaced by the conditions that any finite number of points of X lie in an affine open of X and that the nonregular locus of X, $\mathcal{O}$ is closed. An example is given which shows one difficulty of the nonreduced case.

By a *prime ideal* p *of* $\mathcal{O}$ is meant a subsheaf p of ideals of $\mathcal{O}$ such that for every open $U \subset X$, $\Gamma(U, p)$ is a prime ideal of $\Gamma(U, \mathcal{O})$. These are the points of X by the usual correspondence.

For reducible schemes, the notation for zero-divisors developed in [2] is used. A *zero prime ideal* N of $\mathcal{O}$ is a proper prime ideal of $\mathcal{O}$ such that for each open $U \subset X$, $\Gamma(U, N)$ is either $\Gamma(U, \mathcal{O})$ or consists entirely of zero divisors of $\Gamma(U, \mathcal{O})$. A *divisorial* prime ideal p of $\mathcal{O}$ is a prime ideal p of $\mathcal{O}$ which contains a zero prime ideal N of $\mathcal{O}$ such that p/N is of height one in $\mathcal{O}/N$.

Let $Q\mathcal{O}$ denote the total quotient sheaf of $\mathcal{O}$, and let $K = \Gamma(X, Q\mathcal{O})$. Let $\mathfrak{D} = \Gamma(X, (Q\mathcal{O})^*/\mathcal{O}^*)$, the group of Cartier divisors, let $\mathfrak{P} = \Gamma(X, (Q\mathcal{O})^*)/\Gamma(X, \mathcal{O}^*)$, and let $\mathfrak{C} = \mathfrak{D}/\mathfrak{P}$. ($\mathfrak{P}$ is the set of *principal Cartier divisors*, and $\mathfrak{P}$ defines *linear equivalence* on $\mathfrak{D}$.) Let (f) denote the principal Cartier divisor defined by $f \in K^*$.

Received by the editors January 31, 1973 and, in revised form, May 3, 1974.

AMS (MOS) subject classifications (1970). Primary 14C20; Secondary 14B05, 14C10.

Key words and phrases. Divisors, Cartier divisors, generalized equivalence relation.

Let $\mathfrak{D} \subset X$ be the set of nondivisorial prime ideals p of $\mathcal{O}$ such that if f is a regular element of $\mathcal{O}_p$ then $\mathcal{O}_p p$ is an embedded prime ideal of $\mathcal{O}_p f$. Let $\mathcal{C}$ be the union of $\mathfrak{D}$ and the set of nonnormal (nonregular) divisorial prime ideals. If X, $\mathcal{O}$ is reduced, then $\mathfrak{D}$ is the set of nondivisorial prime ideals of depth (grade) one, and both $\mathfrak{D}$ and $\mathcal{C}$ are finite.

Let D be the group of Weil divisors of X, the free abelian group generated by the divisorial prime ideals of $\mathcal{O}$. Let p be a divisorial prime ideal of $\mathcal{O}$, let $f \in K^*$, let f_1 and f_2 be regular elements of $\mathcal{O}_p$ such that $f_p = f_1/f_2$, and define

$$\lambda_p(f) = \sum_{N \subset p} \left(\ell_{\mathcal{O}_p}\left(\frac{\mathcal{O}_p}{\mathcal{O}_p f_1 + N_p}\right) - \ell_{\mathcal{O}_p}\left(\frac{\mathcal{O}_p}{\mathcal{O}_p f_2 + N_p}\right)\right),$$

where the sum is over all zero prime ideals $N \subset p$ with p/N of height one in $\mathcal{O}/N$. Let the principal Weil divisor defined by $f \in K^*$ be $\langle f \rangle = \Sigma_p \lambda_p(f) p$, where the sum is over all divisorial prime ideals p. Let S be the set of all divisorial prime ideals which are contained in some element of $\mathcal{C}$. Let $D \setminus S$ be the free abelian group generated by those divisorial prime ideals not belonging to S. Let I be the subgroup of D of principal divisors $\langle f \rangle$ where $f \in K^*$ is such that $f_q \in \mathcal{O}_q^*$ for all $q \in \mathcal{C}$, consider I as a subgroup of $D \setminus S$, and let $C = (D \setminus S)/I$. (D/I is the *generalized class group* of X, $\mathcal{O}$ and I defines a *generalized equivalence relation*. Letting P be the subgroup of D of all principal divisors $\langle f \rangle$, D/P is the *class group* of X, $\mathcal{O}$, and P defines *linear equivalence* on D.)

To construct the injection from $\mathfrak{C}$ to C for X, $\mathcal{O}$ reduced, only the following approximation lemma for Cartier divisors is needed. It is a variant of the usual approximation [3, Lemma 3, p. 166].

Recall that if $p_1, \cdots, p_s$ and p are prime ideals of $\mathcal{O}$, then $p \subset p_1 \cup \cdots \cup p_s$ if and only if $p \subset p_i$ for some $i = 1, \cdots, s$. Let $q_1, \cdots, q_t$ also be prime ideals of $\mathcal{O}$. The condition below that

$$p \subset (p_1 \cup \cdots \cup p_s) \cap (q_1 \cup \cdots \cup q_t)$$

is equivalent to $p \subset p_i \cap q_j$ for some i and j.

Lemma. *Let X, $\mathcal{O}$ be a subscheme of a projective space over a field. Let $p_1, \cdots, p_s$ be prime ideals of $\mathcal{O}$, let D be a Cartier divisor on X, and let $q_1, \cdots, q_t$ be prime ideals of $\mathcal{O}$ such that $D_p = (1)_p$ for every prime ideal p of $\mathcal{O}$ with $p \subset (p_1 \cup \cdots \cup p_s) \cap (q_1 \cup \cdots \cup q_t)$. Then there is an*

element f of K^* such that $(f)_{p_i} = D_{p_i}$ for $i = 1, \cdots, s$ and $(f)_{q_i} = (1)_{q_i}$ for $i = 1, \cdots, t$.

Proof. First assume the Lemma is true for $s = 1$, and induct on s. Assuming true for $s - 1$, there is an $f'' \in K^*$ with $(f'')_{q_i} = (1)_{q_i}$ for $i = 1, \cdots, t$ and $(f'')_{p_i} = D_{p_i}$ for $i = 1, \cdots, s - 1$. Then $(D - (f''))_{p_i} = (1)_{p_i}$ for $i = 1, \cdots, s - 1$, and $(D - (f''))_{q_i} = D_{q_i}$ for $i = 1, \cdots, t$. It follows that if p is a prime ideal with

$$p \subset (p_1 \cup \cdots \cup p_{s-1} \cup q_1 \cup \cdots \cup q_t) \cap p_s$$

then $(D - (f''))_p = (1)_p$. Thus there is an $f' \in K^*$ such that $(f')_{p_s} = (D - (f''))_{p_s}$, $(f')_{p_i} = (1)_{p_i}$ for $i = 1, \cdots, s - 1$, and $(f')_{q_i} = (1)_{q_i}$ for $i = 1, \cdots, t$. Let $f = f''/f' \in K^*$.

Now let $s = 1$ and $p = p_1$. Let $A = \Gamma(U, \mathcal{O})$ where U is an affine open subset of X meeting each irreducible component of X and containing the points $q_1, \cdots, q_t$ and p which are to be considered as prime ideals of A. Let g_1 and g_2 be regular elements of A such that $D_p = (g_1/g_2)_p$.

Let $P_1, \cdots, P_r$ be the associated prime ideals of Ag_2 for which $g_1/g_2 \notin A_{P_i}$. P_i is not a zero prime ideal because $g_2 \in P_i$. If $P_i \subset p$ then $D_{P_i} = (g_1/g_2)_{P_i} \neq (1)_{P_i}$ and $P_i \not\subset q_1 \cup \cdots \cup q_t$, and let $e_i \in P_i$ be a regular element of A such that $e_i \notin q_1 \cup \cdots \cup q_t$. Or, if $P_i \not\subset p$, let $e_i \in P_i$ be a regular element of A with $e_i \notin p$. There is an integer $n \geq 1$ such that $(e_1 \cdots e_r)^n g_1 \in Ag_2$. Let $h_1 = (e_1 \cdots e_r)^n g_1 g_2^{-1} \in A$, and let $h_2 = (e_1 \cdots e_r)^n$. Then $D_p = (h_1/h_2)_p$, and no isolated prime ideal of Ah_1 or Ah_2 is contained in $p \cap (q_1 \cup \cdots \cup q_t)$.

Let $Q_1, \cdots, Q_s$ be the associated prime ideals of $A0$ in A. For $j = 1, 2$, let k_j be an element of p contained in all the isolated prime ideals of $A_p h_j$ such that for $i = 1, \cdots, t$, $k_j \in q_i$ if and only if $h_j \notin q_i$, and such that for $i = 1, \cdots, r$, $k_j \in Q_i$ if $Q_i \not\subset q_1 \cup \cdots \cup q_t$. There is an integer $n \geq 1$ such that $k_j^n \in A_p h_j$ for $j = 1, 2$, for k_j is contained in each associated prime ideal of $A_p h_j$. For $j = 1, 2$, let $f_j = h_j + k_j^{n+1}$. Then $f_j \notin q_1 \cup \cdots \cup q_t$, f_j is a regular element of A, $1 + h_j^{-1} k_j^{n+1}$ is a unit of A_p, and $(f_1/f_2)_p = D_p$. Let $f = f_1/f_2$. Q.E.D.

The injection $\phi: \mathfrak{C} \to C$ is to be constructed. Let $\alpha \in \mathfrak{C}$. By the Lemma there is a divisor D in α such that $D_p = (1)_p$ for all $p \in \mathcal{C}$, and therefore $D_p =$

$(1)_p$ for all $p \in S$. For a divisorial prime ideal p, let f_1 and f_2 be regular elements of $\mathcal{O}_p$ with $D_p = (f_1/f_2)_p$, and define

$$\lambda_p(D) = \sum_{N \subset \mathcal{O}_p} \left(\ell_{\mathcal{O}_p}\left(\frac{\mathcal{O}_p}{\mathcal{O}_p f_1 + N}\right) - \ell_{\mathcal{O}_p}\left(\frac{\mathcal{O}_p}{\mathcal{O}_p f_2 + N}\right)\right),$$

where the sum is over all zero prime ideals N of $\mathcal{O}_p$ such that $\mathcal{O}_p\, p/N$ is of height one in $\mathcal{O}_p/N$. Let

$$\lambda D = \sum_{p \notin S} \lambda_p(D) p \in D \backslash S,$$

where the sum is over all divisorial prime ideals p not contained in S. If $f \in K^*$ and $(f)_p = (1)_p$ for all $p \in \mathcal{C}$, then $\lambda(f) = \langle f \rangle \in I$. Letting $\phi\alpha$ be the image of λD in $C = (D \backslash S)/I$, $\phi: \mathfrak{C} \to C$ is a well-defined homomorphism.

Theorem. *Let X, $\mathcal{O}$ be a reduced subscheme of a projective space over a field. The length homomorphism $\phi: \mathfrak{C} \to C$ is injective. The image of ϕ is the subgroup of locally principal elements of C, which is a subquotient of D. Furthermore $\phi: \mathfrak{C} \to C$ is surjective if and only if $\mathcal{O}_p$ is factorial for all prime ideals p of $\mathcal{O}$ which are contained in no element of $\mathcal{C}$.*

Proof. The usual argument follows ([4, Propositions, pp. 65, 66]).

Let $\mathfrak{D} \backslash \mathcal{C}$ be the subgroup of $\mathfrak{D}$ of divisors D such that $D_p = (1)_p$ for all $p \in \mathcal{C}$. Let $D \in \mathfrak{D} \backslash \mathcal{C}$ with $\lambda D = \langle 1 \rangle$. Let p be a prime ideal of $\mathcal{O}$, and let f_1 and f_2 be regular elements of $\mathcal{O}_p$ such that $D_p = (f_1/f_2)_p$. If $q \subset p$ is a prime ideal contained in $\mathcal{C}$, then $D_q = (1)_q$ and $\mathcal{O}_q f_1 = \mathcal{O}_q f_2$. If $q \subset p$ is a divisorial prime ideal not contained in $\mathcal{C}$, then $\ell_{\mathcal{O}_q}(\mathcal{O}_q/\mathcal{O}_q f_1) = \ell_{\mathcal{O}_q}(\mathcal{O}_q/\mathcal{O}_q f_2)$ and, because $\mathcal{O}_q$ is normal of Krull dimension one, $\mathcal{O}_q f_1 = \mathcal{O}_q f_2$. Thus $\mathcal{O}_q f_1 = \mathcal{O}_q f_2$ for all depth one (grade one) prime ideals q of $\mathcal{O}_p$, and

$$\mathcal{O}_p f_1 = \bigcap_q \mathcal{O}_q f_1 = \bigcap_q \mathcal{O}_q f_2 = \mathcal{O}_p f_2.$$

Hence $\mathcal{O}_p f_1 = \mathcal{O}_p f_2$ for all prime ideals p of $\mathcal{O}$, $f_1/f_2 \in \mathcal{O}_p^*$, $D = (1)$, and λ: $\mathfrak{D} \backslash \mathcal{C} \to D \backslash S$ is injective.

Now, to show that $\phi: \mathfrak{C} \to C$ is injective, let $\alpha \in \mathfrak{C}$ with $\phi\alpha = 0 \in C$, and let $D \in \alpha$ be such that $D_q = (1)_q$ for all q in $\mathcal{C}$. Let $f \in K^*$ be such that $f \in \mathcal{O}_q^*$ for all q in $\mathcal{C}$ and $\lambda D = \langle f \rangle$. Then by the injectivity of λ above, $D = (f)$, and $\alpha = 0$.

$\phi: \mathfrak{C} \to C$ is surjective if and only if $\lambda: \mathfrak{D} \backslash \mathcal{C} \to D \backslash S$ is surjective. This is true if and only if the group of Cartier divisors is equal to the group

of Weil divisors for each local ring $\mathcal{O}_p$ where p is a prime ideal of $\mathcal{O}$ which is contained in no element of $\mathcal{C}$, which is in turn equivalent to each $\mathcal{O}_p$ being factorial for all these prime ideals p. Q.E.D.

If $\mathcal{O}$ is not reduced, $\mathcal{C}$ may no longer be finite, and the construction used to define ϕ may not be applicable. For example, let $R = k[x, y, z]/(x^2, xy)$ where k is a field. The reduction of R, $R/Rx = k[y, z]$, is the polynomial ring in two variables over k. Let p be a prime element of $k[z]$. Rp has as associated prime ideals (p, x) and (p, x, y), for (x, y) is an embedded component of (0) in

$$(k[z]/(p))[x, y]/(x^2, xy) \simeq R/Rp.$$

Thus $(p, x, y) \in \mathfrak{D}$, and $\mathfrak{D}$ is infinite. The similar projective example given by the homogeneous ring $k[W, X, Y, Z]/(X^2, XY)$ is such that every homogeneous height one prime ideal is contained in an element of $\mathfrak{D}$.

REFERENCES

1. Séminaire C. Chevalley, 3ième année: 1958/59, *Variétés de Picard*, Ecole Normal Supérieure, Paris, 1960. MR 28 #1094.

2. J. Hornell, *Divisorial complete intersections*, Pacific J. Math. 45 (1973), 217–227. MR 47 #6668.

3. S. Lang, *Abelian varieties*, Interscience Tracts in Pure and Appl. Math., no. 7, Interscience, New York, 1959. MR 21 #4959.

4. D. Mumford, *Lectures on curves on an algebraic surface*, Ann. of Math. Studies, no. 59, Princeton Univ. Press, Princeton, N. J., 1966. MR 35 #187.

5. M. Rosenlicht, *Equivalence relations on algebraic curves*, Ann. of Math. (2) 56 (1952), 169–191. MR 14, 80.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66044

Current address: 2017 North 6th Street Terrace, Blue Springs, Missouri 64015