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SCALAR AND VECTOR VALUED PREMEASURES

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ABSTRACT. A real valued (respectively Banach space valued) set-function on a lattice of sets extends to a σ-additive measure on a σ-field provided it is finitely additive tight, continuous at $\emptyset$ and has a bounded (respectively conditionally weakly compact) range.

In an earlier paper [2] with J. L. Kelley, we showed that a nonnegative valued function μ on a lattice $\mathfrak{A}$ of sets is a *premeasure*, meaning that it extends to a countably additive measure on a δ-ring $\supset \mathfrak{A}$, provided μ is tight and continuous at $\emptyset$. The extension of the theorem to the case of real valued μ (not necessarily nonnegative valued) bugged us for sometime. The real valued case differs from the nonnegative case in that in the latter, for each monotonic increasing sequence $\{A_n\}$ of members of $\mathfrak{A}$, the sequence $\{\mu(A_n)\}_n$ is monotonic increasing and therefore has a limit, while for arbitrary real values, the existence of limits for such sequences is far from being clear. We show in this paper that the theorem indeed is true in the real valued case as well. An obvious additional hypothesis is necessary, namely that μ is locally bounded, i.e. $\forall A \in \mathfrak{A}$, $\operatorname{Sup}\{\mu(B)/B \in \mathfrak{A},\ B \subseteq A\} < \infty$. (This is trivially true in case the values of μ are nonnegative.) Explicitly we have the following.

Theorem 1. *A real valued μ on a lattice $\mathfrak{A}$ of sets extends to a countably additive measure on a δ-ring (respectively σ-field) containing $\mathfrak{A}$ provided μ is*

(i) *locally bounded (respectively bounded),*

(ii) *finitely additive,*

(iii) *continuous at $\emptyset$,*

(iv) *tight.*

In the above theorem we use a weaker definition for tightness than in

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[2], keeping in mind its adaptation to the vector valued case. When μ is nonnegative valued, conditions (i), (ii) and (iv) are all implied by the form of tightness assumed in [2].

The main point of the proof of Theorem 1 is that μ decomposes as usual as a difference, $\mu^+ - \mu^-$, of nonnegative valued functions, where both μ^+ and μ^- preserve all the properties of μ: finite additivity, continuity at $\emptyset$, and tightness. This is established in Steps I–V below. Then the nonnegative valued case of the theorem proved in [2] applies to μ^+ and μ^- and yields the theorem for μ.

From the real valued case, a corresponding result follows for functions μ with values in a Banach space with or without the use of known theorems on extension [3]. The precise theorem in case of Banach space valued μ is as follows:

Theorem 2. *A Banach space valued μ on a lattice $\mathfrak{A}$ of sets extends to a countably additive measure on a σ-field containing $\mathfrak{A}$ with values in the Banach space provided μ is*

(i) *finitely additive,*

(ii) *continuous at $\emptyset$,*

(iii) *tight, and*

(iv) *$\mu[\mathfrak{A}]$ is conditionally weakly compact.*

We outline the proofs below. We need to spell out some definitions. Let μ be defined on a family $\mathfrak{A}$ of sets with values in a Banach space E; we always assume that $\emptyset \in \mathfrak{A}$ and $\mu(\emptyset) = 0$.

μ is said to be *continuous at* $\emptyset$ iff for each decreasing sequence $\{A_n\}_n$ of members of $\mathfrak{A}$ with $\bigcap_n A_n = \emptyset$, we have $\lim_n \mu(A_n) = 0$.

μ is said to be *tight* iff for every pair of sets A', A'' in $\mathfrak{A}$ such that $A' \supseteq A''$, and for arbitrary $\epsilon > 0$ there exists a set $A \in \mathfrak{A}$, $A \subseteq A' - A''$ such that $\|\mu(A') - \mu(A'') - \mu(A)\| < \epsilon$. (If the values of μ are real or complex scalars, the norm above is, of course, the absolute value.)

Proof of Theorem 1. Let μ^+, μ^- be defined on $\mathfrak{A}$ by

$$\mu^+(A') = \operatorname{Sup}\{\mu(A)/A \subseteq A',\ A \in \mathfrak{A}\}$$

and

$$\mu^-(A') = -\inf\{\mu(A)/A \subseteq A',\ A \in \mathfrak{A}\} \quad \text{for any } A' \in \mathfrak{A}.$$

Then μ^+ and μ^- are finite and nonnegative because μ is assumed to be locally bounded. The proof is organized in the form of Steps I–V below. We note that $\mu^- = (-\mu)^+$. Consequently, the assertions on μ^- in the following steps follow from the corresponding facts for μ^+.

Step I. $\mu = \mu^+ - \mu^-$.

Proof. Let $\epsilon > 0$ and let A be any set in $\mathfrak{A}$. Then there exists $A_1 \in \mathfrak{A}$, $A_1 \subseteq A$ such that

$$\mu^+(A) \leq \mu(A_1) + \epsilon/2. \tag{I. 1}$$

Since μ is tight, there exists $A_2 \in \mathfrak{A}$, $A_2 \subseteq A - A_1$ such that

$$\mu(A_1) + \mu(A_2) \leq \mu(A) + \epsilon/2. \tag{I. 2}$$

By (I.1) and (I.2)

$$\begin{aligned} \mu^+(A) - \mu^-(A) &\leq \mu(A_1) + \mu(A_2) + \epsilon/2 \quad (\text{since } -\mu^-(A) \leq \mu(A_2)) \\ &\leq \mu(A) + \epsilon. \end{aligned} \tag{I. 3}$$

Applying (I.3) to $-\mu$ instead of μ, we get

$$\mu^+(A) - \mu^-(A) \geq \mu(A) - \epsilon. \tag{I. 4}$$

Hence the result follows from (I.3) and (I.4).

Step II. μ^+, μ^- are finitely additive.

Proof. Let $\epsilon > 0$ be chosen. Given any two disjoint sets A_1, A_2 in $\mathfrak{A}$, by a suitable choice of $A_{11}, A_{21} \in \mathfrak{A}$ such that $A_{11} \subseteq A_1$, $A_{21} \subseteq A_2$, we have

$$\begin{aligned} \mu^+(A_1) + \mu^+(A_2) &\leq \mu(A_{11}) + \mu(A_{21}) + \epsilon \\ &= \mu(A_{11} \cup A_{21}) + \epsilon \leq \mu^+(A_1 \cup A_2) + \epsilon. \end{aligned}$$

On the other hand there exists $\overline{A} \subseteq A_1 \cup A_2$ such that

$$\begin{aligned} \mu^+(A_1 \cup A_2) \leq \mu(\overline{A}) + \epsilon &= \mu(\overline{A} \cap A_1) + \mu(\overline{A} \cap A_2) + \epsilon \\ &\leq \mu^+(A_1) + \mu^+(A_2) + \epsilon. \end{aligned}$$

It follows that $\mu^+(A_1 \cup A_2) = \mu^+(A_1) + \mu^+(A_2)$.

Step III. μ^+, μ^- are tight.

Proof. Consider any pair of sets in $\mathfrak{A}$ such that one contains the other, say $A_1, A_2 \in \mathfrak{A}$, $A_1 \supseteq A_2$. Given $\epsilon > 0$, there exists $A_3 \in \mathfrak{A}$ such that $A_3 \subseteq A_1 - A_2$ and

$$\operatorname{Sup}\{\mu(A)/A \subseteq A_1 - A_2\} \leq \mu(A_3) + \epsilon/3. \tag{III. 1}$$

This implies that any set A disjoint with A_3 and lying in $A_1 - A_2$ cannot have a measure larger than $\epsilon/3$. Otherwise $A \cup A_3$ will have a measure larger than the supremum on the left-hand side of (1), which is impossible.

Let $A_4 \subseteq A_1$, $A_4 \in \mathfrak{A}$, be so chosen that

$$\begin{aligned}\mu^+(A_1) &\leq \mu(A_4) + \epsilon/3 \\ &= \mu(A_4 \cap (A_2 \cup A_3)) + (\mu(A_4) - \mu(A_4 \cap (A_2 \cup A_3))) + \epsilon/3.\end{aligned} \tag{III. 2}$$

By tightness we can choose $A \in \mathfrak{A}$ contained in $A_4 - (A_4 \cap (A_2 \cup A_3))$ so that $\mu(A)$ differs from $\mu(A_4) - \mu(A_4 \cap (A_2 \cup A_3))$ by less than $\epsilon/3$. By our remark above, $\mu(A) \leq \epsilon/3$. It follows that the difference $\mu(A_4) - \mu(A_4 \cap (A_2 \cup A_3))$ is itself less than $2\epsilon/3$. Consequently from (III.2),

$$\begin{aligned}\mu^+(A_1) &< \mu(A_4 \cap (A_2 \cup A_3)) + \epsilon \\ &\leq \mu^+(A_2 \cup A_3) + \epsilon = \mu^+(A_2) + \mu^+(A_3) + \epsilon;\end{aligned}$$

$$\mu^+(A_1) - \mu^+(A_2) < \mu^+(A_3) + \epsilon.$$

The reverse inequality is obvious because μ^+ is monotone and finitely additive. Tightness of μ^+ (and therefore of μ^-) follows.

Step IV. μ is modular; i.e. for every pair $A_1, A_2 \in \mathfrak{A}$

$$\mu(A_1 \cup A_2) + \mu(A_1 \cap A_2) = \mu(A_1) + \mu(A_2).$$

Proof. Since $\mu = \mu^+ - \mu^-$, it suffices to check that μ^+ (and therefore μ^-) is modular. In turn (see [1]) it suffices to show that for $A_1, A_2 \in \mathfrak{A}$, $A_1 \supseteq A_2$, $\mu^+(A_1) - \mu^+(A_2)$ depends only on the difference $A_1 - A_2$. This is immediate because, as a result of tightness and monotonicity of μ^+,

$$\mu^+(A_1) - \mu^+(A_2) = \operatorname{Sup}\{\mu^+(A)/A \subseteq A_1 - A_2,\ A \in \mathfrak{A}\}.$$

Step V. μ^+, μ^- are continuous at $\emptyset$.

Proof. Let $\{A_n\}_n$ be a decreasing sequence in $\mathfrak{A}$ such that $\bigcap^\infty A_n = \emptyset$. Let $\epsilon > 0$ be chosen. A sequence $\{B_n\} \subset \mathfrak{A}$, $B_n \subseteq A_n$, $\forall n$ can be chosen such that

$$\mu^+(A_n) \leq \mu(B_n) + \epsilon/2^{n+1}. \tag{V. 1}$$

This means that any set contained in A_n and disjoint with B_n cannot have a measure larger than $\epsilon/2^{n+1}$.

For any integer n and any $k < n$, we can choose, by tightness, an $A \in \mathfrak{A}$, A contained in

$$((B_n \cap (B_1 \cap \cdots \cap B_{k-1})) \cup B_k) - B_k$$

so that $\mu(A)$ differs from

$$\mu((B_n \cap (B_1 \cap B_2 \cap \cdots \cap B_{k-1})) \cup B_k) - \mu(B_k)$$

by at most $\epsilon/2^{k+1}$. By our remark above, $\mu(A) \leq \epsilon/2^{k+1}$. It follows that for any n and $k < n$

$$\mu((B_n \cap (B_i \cap \cdots \cap B_{k-1})) \cup B_k) - \mu(B_k) \leq \epsilon/2^k. \tag{V. 2}$$

Then we claim that for all m,

$$\mu^+(A_n) \leq \mu(B_1 \cap \cdots \cap B_n) + \epsilon/2 + \cdots + \epsilon/2^n. \tag{V. 3}$$

For $n = 1$, this immediately follows from (V.1). Next,

$$\mu^+(A_2) \leq \mu(B_2 \cap B_1) + \mu(B_2 \cup B_1) - \mu(B_1) + \epsilon/2^2$$

(because of (V.1) and modularity of μ)

$$\leq \mu(B_1 \cap B_2) + \epsilon/2 + \epsilon/2^2 \quad \text{(by (V.2))}.$$

Similarly for $n = 3$,

$$\begin{aligned}
\mu^+(A_3) &\leq \mu(B_3 \cap B_1) + \mu(B_3 \cup B_1) - \mu(B_1) + \epsilon/2^3 \\
&\leq \mu(B_3 \cap B_1) + \epsilon/2 + \epsilon/2^3 \quad \text{(by (V.2))} \\
&= \mu(B_3 \cap B_1 \cap B_2) + \mu((B_3 \cap B_1) \cup B_2) - \mu(B_2) + \epsilon/2 + \epsilon/2^3 \\
&\qquad\qquad \text{(because } \mu \text{ is modular)} \\
&\leq \mu(B_1 \cap B_2 \cap B_3) + \epsilon/2 + \epsilon/2^2 + \epsilon/2^3 \quad \text{(by (V.2))}.
\end{aligned}$$

Continuing the above process, we see that (V.3) holds good for all values of n. As $\bigcap_n B_n = \varnothing$ and μ is continuous at $\varnothing$, it follows from (V.3) that $\lim_n \mu^+(A_n) = 0$.

We have shown that μ^+ and μ^- satisfy the conditions for a premeasure set out in [2]. Further, they can be extended as measures on the σ-field generated by $\mathfrak{A}$ in case μ is bounded instead of being locally bounded. The corresponding conclusion for μ follows.

We can derive Theorem 2 fromTheorem 1 using direct arguments. But we shall make use of a known extension theorem [3] and argue as follows. For each x^* in E^*, the scalar function $x^*\mu$ is bounded, finitely additive, continuous at $\varnothing$, and tight, and, therefore, can be extended as a scalar measure $\widehat{x^*\mu}$ on the ring $\mathfrak{R}$ generated by $\mathfrak{A}$. Each member of $\mathfrak{R}$ is a finite disjoint union $\bigcup_{i=1}^n (A_i - B_i)$, where for each i, $A_i, B_i \in \mathfrak{A}$, and $A_i \supset B_i$. For each x^* in E^* we have

$$x^* \left(\sum_i [\mu(A_i) - \mu(B_i)] \right) = \widehat{x^*\mu} \left(\bigcup_i A_i \setminus B_i \right),$$

which shows that if we let

$$\bar{\mu}\left(\bigcup_{i=1}^{n} (A_i - B_i)\right) = \sum_{i=1}^{n} (\mu(A_i) - \mu(B_i)),$$

then $\bar{\mu}$ is a well-defined weakly countably additive function on $\mathfrak{R}$ extending μ. Using tightness we see that $\mu[\mathfrak{A}]$ is dense in $\bar{\mu}[\mathfrak{R}]$ so that the latter is still weakly conditionally compact. Now the known theorem on extension applies to $\bar{\mu}$ (see [3, p. 178]) and completes the proof.

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