AN EXTENDED INEQUALITY FOR
THE MAXIMAL FUNCTION

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ABSTRACT. Fefferman and Stein [3] have proved an \( L^p \) inequality for the Hardy-Littlewood maximal function applied to functions taking values in a sequence space \( l^p \). This note extends their theorem to functions taking values in a mixed \( L^p \) space. An application to mixed estimates for Riesz potentials is given.

1. Notation. Let \( k = (k_1, \ldots, k_n) \) be an \( n \)-tuple of natural numbers, and let \( P = (p_1, \ldots, p_n) \), where \( 1 \leq p_i \leq \infty \). For \( f = \{f_k\}_{k \in \mathbb{N}^n} \), \( K_p(f) \) is defined by successively computing the \( l^{p_i} \) norm with respect to \( k_i \) while \( k_{i+1}, \ldots, k_n \) are held fixed. The set of sequences on \( \mathbb{N}^n \) for which this norm is finite is denoted by \( l^P \).

For \( \phi \) a complex-valued locally integrable function on \( R^m \), \( \phi^* \) denotes the maximal function

\[
\phi^*(x) = \sup_{r > 0} \frac{1}{mB(x, r)} \int_{B(x, r)} |\phi(y)| dy.
\]

Here \( B(x, r) \) is the ball with center \( x \) and radius \( r \); \( mB(x, r) \) is its measure.

For \( f = \{f_k\} \) a function on \( R^m \) with values in \( l^P \), \( f^* \) is the \( l^P \)-valued function obtained by taking the maximal function of each \( f_k \).

2. Theorem. For \( 1 < p_i < \infty \) and \( 1 < q < \infty \), there is a constant \( c \) such that

\[
\int K_p(f^*)^q dx \leq c \int K_p(f)^q dx.
\]

Proof. We perform induction on \( n \). Fefferman and Stein [3] have treated the case \( n = 1 \); we could just as well start with \( n = 0 \). Assume the Theorem has been established for some fixed value of \( n \). We show that for \( f = \{f_{j,k}\}_{j \in \mathbb{N}, k \in \mathbb{N}^n} \) and \( 1 < r < \infty \),
\[
\int K_P \left[ \left( \sum_{j=1}^{\infty} |f_{j,k}^*|^r \right)^{1/r} \right]^q dx \leq c \int K_P \left[ \left( \sum_{j=1}^{\infty} |f_{j,k}^*|^r \right)^{1/r} \right]^q dx
\]

or, more briefly,

\[(*) \quad \int K_P J_r (f^*)^q dx \leq c \int K_P J_r (f)^q dx.\]

Note that for each \( j, k \) we have

\[|f_{j,k} (x)| \leq \sup_j |f_{j,k} (x)| = J_{\infty} (f_{j,k}) (x).\]

Thus \( f_{j,k} (x) \leq J_{\infty} (f_{j,k}) (x) \), and so \( J_{\infty} (f_{j,k}) \leq J_{\infty} (f_{j,k})^* \). Hence

\[\int K_P J_{\infty} (f^*)^q dx \leq \int K_P [J_{\infty} (f)]^q dx \leq c \int K_P [J_{\infty} (f)]^q dx\]

by the inductive hypothesis. Thus \((*)\) is valid for \( r = \infty \).

Now suppose \( 1 < r < \min (q, p_1, \ldots, p_n) \). Note that

\[K_P J_r (f^*) = K_P \left[ \left( \sum_{j=1}^{\infty} |f_{j,k}^*|^r \right)^{1/r} \right]^{1/r} = K_{P/r} \left[ \sum_{j=1}^{\infty} |f_{j,k}^*|^r \right]^{1/r},\]

where \( P/r = (p_1/r, p_2/r, \ldots, p_n/r) \). Thus by the duality established in Benedek and Panzone [2],

\[\int K_P J_r (f^*)^q dx = \int K_{P/r} \left[ \sum_{j=1}^{\infty} |f_{j,k}^*|^r \right]^{q/r} dx = \sup_\phi \int \sum_{k} \left[ \sum_{j=1}^{\infty} |f_{j,k}^*|^r \right] \phi_k dx \]

where the supremum is taken over all \( \phi = \{\phi_k\} \) for which \( \int K_{(P/r)^*} (\phi)^{q/(q-r)} dx \leq 1 \). By Lemma 1 of [3],

\[\left| \sum_k \left[ \sum_{j=1}^{\infty} |f_{j,k}^*|^r \right] \phi_k dx \right| \leq \sum_k \sum_{j=1}^{\infty} |f_{j,k}^*|^r |\phi_k| dx \leq c \sum_k \sum_{j=1}^{\infty} |f_{j,k}^*|^r \phi_k^* dx \]

\[\leq \int K_{P/r} \left[ \sum_{j=1}^{\infty} |f_{j,k}^*|^r \right] K_{(P/r)^*} [\phi^*] dx \]

\[\leq \left( \int K_{P/r} \left[ \sum_{j=1}^{\infty} |f_{j,k}^*|^r \right]^{q/r} dx \right)^{r/q} \cdot \left( \int K_{(P/r)^*} [\phi^*]^{q/(q-r)} dx \right)^{1-r/q}\]
using successive applications of Hölder’s inequality for sequences and integrals. By the inductive hypothesis
\[ \int K_{(P/r)}[|\phi|^q/(q-r)] dx \leq c \int K_{(P/r)}[|\phi|^q/(q-r)] dx \leq c. \]

Since \( \int K_{p/r} \left[ \sum_{j=1}^{\infty} |f_j|^r \right]^{q/r} dx = K_p f_q dx \), this yields (*)

We complete the proof by an interpolation process. Benedek and Panzone [2] give an extension of the Riesz-Thorin interpolation theorem to mixed \( L^p \) spaces. However, the theorem applies directly only to linear operators, and the operator under consideration is nonlinear.

Instead we look at a linear operator \( T \) defined by
\[ (Tf)_{i,j,k}(x) = \frac{1}{mB(x, 2^i)} \int_{B(x, 2^i)} f_{i,j,k}(x) dx. \]
Obviously
\[ l_\infty[(Tf)_{i,j,k}(x)] = \sup_{-\infty < i < \infty} |(Tf)_{i,j,k}(x)| \leq f_{i,j,k}^*(x) \]
while \( f_{i,j,k}^*(x) \leq 2^m l_\infty[(Tf)_{i,j,k}(x)] \). (Here \( |f| = \{|f_{i,j,k}| \} \). Thus, (*) holds for all \( f \) if and only if
\[ \int K_p l_\infty(Tf)^q dx \leq c \int K_p f_q dx \]
for all \( f \).

We have seen that (*) and, hence, (**) is valid for \( r = \infty \) and for \( 1 < r < \min(q, \frac{1}{p_1}, \cdots, \frac{1}{p_n}) \); interpolation yields (**) for \( 1 < r \leq \infty \). Thus, (*) is valid for \( 1 < r \leq \infty \) and the theorem is proved.

3. A generalization. There is no particular difficulty in replacing the discrete variable \( k \) by a continuous variable \( t \). Let \( (\Omega_i, \mu_i) \) be \( \sigma \)-finite measure spaces, and let \( t = (t_1, \cdots, t_n) \in \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n = \Omega \). For \( f(x, t) \) a locally integrable function on \( R^m \times \Omega \), let
\[ f^*(x, t) = \sup_{r > 0} \frac{1}{mB(x, r)} \int_{B(x, r)} |f(y, t)| dy. \]
Then we have \( \int T_p(f^*)^q dx \leq c \int T_p f^q dx \), where \( T_p \) denotes the mixed \( L^p \) norm taken with respect to \( t \).

4. An application. In [1], mixed norm estimates are obtained for Riesz potentials. One technique used there involves estimates in terms of a maximal function taken with respect to one of the variables. Then the proof is completed via the Calderón-Zygmund theory of singular integrals.
alternate approach would be to use the Theorem stated above. This method would improve the exponent in Theorem 3 and remove the restriction \( p_{l+1} \geq p_{l+2} \geq \cdots \geq p_n \) in Theorem 3' of [1].

REFERENCES


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