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A REPRESENTATION OF GENERALIZED INFRAPOLYNOMIALS[1]

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ABSTRACT. A representation theorem is obtained for generalized infrapolynomials on an arbitrary complex point set. Some results on the location of zeros are also presented.

Let P_n denote the class of polynomials $\Sigma_{i=0}^n c_i z^i$ with complex coefficients considered as mappings of the complex z-plane $\mathcal{C}_z$ into itself. Let $\{\mathfrak{L}^i\}_{i=1}^s$ denote a fixed set of s linearly independent linear functionals on P_n, and $A = A_1, A_2, \cdots, A_s$ be a fixed s-tuple of complex numbers. Then $P_n(A)$ will represent the class of polynomials $p(z)$ in P_n such that $\mathfrak{L}^i p = A_i$, $i = 1, 2, \cdots, s$. Further, let E denote a compact subset of $\mathcal{C}_z$ containing at least $n - s + 2$ points. As in [1] we make the following

Definition. $p(z) \in P_n(A)$ is called an infrapolynomial on E with respect to $P_n(A)$ if $p(z)$ has on E no underpolynomials in $P_n(A)$; i.e., if there exists no polynomial $q(z)$ in $P_n(A)$ such that

(1) $|q(z)| < |p(z)|$ on $E \cap \{z; p(z) \neq 0\}$,

(2) $q(z) = 0$ on $E \cap \{z; p(z) = 0\}$.

A polynomial $q(z) \in P_n(A)$ such that

(3) $|q(z)| \leq |p(z)|$ on E

is called a weak underpolynomial of $p(z)$ on E with respect to $P_n(A)$.

Let e_z denote the functional which gives point evaluation at z, i.e. $e_z f = f(z)$.

Haar Assumption I. $\{\mathfrak{L}^i\}_{i=1}^s \cup \{e_{z_j}\}_{j=s+1}^{n+1}$ is a linearly independent set of functionals on P_n for all choices of $n - s + 1$ distinct points $z_{s+1}, \cdots, z_{n+1}$ in E.

Note 1. Recall that, by assumption, E contains at least $n - s + 2$ points. Note that if E were to contain only $n - s + 1$ points, $P_n(A)$ would contain a unique infrapolynomial, vanishing of course at all the points of E.

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Note also that if E were to contain $n - k + 1$ points, $k \geq s$, $P_n(A)$ would contain a $(k - s)$-dimensional set of infrapolynomials of the form $I_0(z) + h(z)$, where I_0 is a fixed element of $P_n(A)$ vanishing on E, and $h(z)$ is an arbitrary element of P_n vanishing on E and such that $\mathfrak{L}^i h = 0$ $(i = 1, \cdots, s)$.

Note 2. In [1] it was shown that under a slightly stricter assumption (namely the Lagrange set $\{e_{z_j}\}_{j=s+1}^{n+1}$ on $n - s + 1$ points is replaced by the more general Hermite set of $n - s + 1$ derivative evaluations in Haar Assumption I), an infrapolynomial on E with respect to $P_n(A)$ has no weak underpolynomials on E with respect to $P_n(A)$.

From Theorems 1, 2 and 3 of [4] we immediately obtain the following result.

Theorem 1. *If, with respect to* $P_n(A)$, $p(z)$ *is an infrapolynomial on* E, *then* $p(z)$ *is an infrapolynomial on a set* $E_r = \{z_s, z_{s+1}, \cdots, z_{n+r+1}\} \subset E$, *where* $0 \leq r \leq n - s + 1$. *In particular, if the elements of* $P_n(A)$ *are all real-valued on* E, *then we may take* $r = 0$.

Haar Assumption II. Given E_r as above, $\{\mathfrak{L}^i\}_{i=1}^{s-1} \cup \{e_{z_j}\}_{j=s}^{n+r+1}$ is a linearly independent set of functionals on P_{n+r}.

Note 3. In this assumption $\mathfrak{L}^s$ has been removed from consideration.

We will next give our main result which is a representation of infrapolynomials on E_r with respect to $P_n(A)$. But first we introduce some notation. Let $\{e_i\}_{i=1}^{n+r+1}$ represent the set of polynomials in P_{n+r} dual to the set of linear functionals

$$\{M^i\}_{i=1}^{n+r+1} = \{\mathfrak{L}^i\}_{i=1}^{s-1} \cup \{e_{z_j}\}_{j=s}^{n+r+1},$$

i.e., $M^i e_j = \delta_{ij}$ (kronecker delta) $(i, j = 1, 2, \cdots, n + r + 1)$. For each $j = 1, 2, \cdots, n + r + 1$, let $\tilde{e}_j = e_j/\mathfrak{L}^s e_j$ if $\mathfrak{L}^s e_j \neq 0$ and let $\tilde{e}_j = e_j$ if $\mathfrak{L}^s e_j = 0$. Let $\lambda = (\lambda_s, \lambda_{s+1}, \cdots, \lambda_{n+r+1})^T$ be a column vector of $n + r - s + 2$ complex numbers and $\|\lambda\| = (|\lambda_s|, \cdots, |\lambda_{n+r+1}|)$. We introduce the partial ordering "$\leq$" defined by $\|\lambda^*\| \leq \|\lambda\|$ provided $|\lambda_j^*| \leq |\lambda_j|$, $s \leq j \leq n + r + 1$, and $\|\lambda^*\| < \|\lambda\|$ provided $\lambda \neq 0$ and either $\lambda_j^* = 0$ or $|\lambda_j^*| < |\lambda_j|$, $s \leq j \leq n + r + 1$. Let Q be the unique polynomial in P_{n+r} guaranteed by Haar Assumption II such that $\mathfrak{L}^i Q = A_i$ $(i = 1, 2, \cdots, s - 1)$ and $Q(z_i) = 0$ $(i = s, s + 1, \cdots, n + r + 1)$. Finally, let F be the $(r + 1) \times (n + r - s + 2)$-matrix where $F_{ij} = \tilde{e}_j^{(n+i)}(0)$ for $i = 1, \cdots, r$ and $j = s, \cdots, n + r + 1$ and $F_{r+1,j} = \mathfrak{L}^s \tilde{e}_j$, $j = s, \cdots, n + r + 1$, and let G be the $(r + 1)$-column vector

$$(-Q^{(n+1)}(0), \cdots, -Q^{(n+r)}(0), A_s - \mathfrak{L}^s Q)^T.$$

Theorem 2. *The set of infrapolynomials on* E_r *with respect to* $P_n(A)$ *is*

$$\left\{Q + \sum_{j=s}^{n+r+1} \lambda_j^* \tilde{e}_j;\ \lambda^* \textit{ yields } \min_{F\lambda=G} \|\lambda\|\right\}.$$

Note 4. The condition $F\lambda = G$ is the requirement that $R_\lambda = Q + \Sigma_{j=s}^{n+r+1}\lambda_j\tilde{e}_j \in P_n$ and that $\mathfrak{L}^s R_\lambda = A_s$. (Note that by virtue of $M^i e_j = \delta_{ij}$, $\mathfrak{L}^i R_\lambda = A_i$ $(i = 1, 2, \cdots, s-1)$.)

Proof of Theorem 2. If $T \in P_n(A)$, consider $T^* = T - Q \in P_{n+r}$. Then $\mathfrak{L}^i T^* = 0$ $(i = 1, 2, \cdots, s-1)$. Thus

$$T^* = \sum_{j=1}^{n+r+1} (M^j T^*) e_j = \sum_{j=s}^{n+r+1} T^*(z_j)\gamma_j \tilde{e}_j = \sum_{j=s}^{n+r+1} \lambda_j^* \tilde{e}_j,$$

where $\gamma_j = \mathfrak{L}^s e_j$ or $\gamma_j = 1$ and $\lambda_j^* = T(z_j)\gamma_j$. Furthermore, clearly $F\lambda^* = G$. Thus $T = Q + \Sigma_{j=s}^{n+r+1} \lambda_j^* \tilde{e}_j$, where $F\lambda^* = G$, indicates an arbitrary element of $P_n(A)$. Since $Q(z_i) = 0$ $(i = s, s+1, \cdots, n+r+1)$, T is an infrapolynomial precisely when λ^* yields $\min_{F\lambda=G}\|\lambda\|$. This follows from the definition of an infrapolynomial on E_r with respect to $P_n(A)$, the definition of $\|\cdot\|$, and the partial ordering. □

We will give an example of Theorem 2 in the case $r > 0$ (see Example 5), but first we restate this result in the case $r = 0$ and then observe that we obtain several known results as special cases of the case $r = 0$. Let $\beta = A^s - \mathfrak{L}^s Q$ and let $\Sigma'\alpha_j = \Sigma_{\mathfrak{L}^s e_j \neq 0}\, \alpha_j$.

Corollary 1. *The set of infrapolynomials on* E_0 *with respect to* $P_n(A)$ *is*

$$(*)\qquad \left\{Q + \beta\sum_{j=s}^{n+1} \lambda_j' \tilde{e}_j;\ \sum{}'\lambda_j' = 1 \textit{ wherein } \lambda_j' \geq 0\right\}.$$

Proof. In case $r = 0$, $R_\lambda \in P_n$ and $F\lambda = G$ is the single requirement that

$$\mathfrak{L}^s R_\lambda = \mathfrak{L}^s Q + \sum_{j=s}^{n+r+1} \lambda_j \mathfrak{L}^s \tilde{e}_j = \mathfrak{L}^s Q + \sum_{j=s}^{n+r+1}{}' \lambda_j = \mathfrak{L}^s Q + \beta,$$

i.e. that $\Sigma'\lambda_j = \beta$. Then $\min\|\lambda\|$ is yielded precisely whenever $\operatorname{sgn}\lambda_j = \operatorname{sgn}\beta$ for all λ_j occurring in $\Sigma'\lambda_j$, for otherwise $\tilde{\lambda} < \lambda$, where $\tilde{\lambda}_j = |\lambda_j|\beta/\Sigma|\lambda_i|$. The conclusion follows by setting $\lambda_j' = \lambda_j/\beta$, if $\beta \neq 0$. □

Example 1 (Fekete [2], Motzkin-Walsh [3]). Let $E = \{z_1, \cdots, z_{n+1}\}$ and $g(z) = \Pi_{i=1}^{n+1}(z - z_i)$. Then the set of infrapolynomials on E with leading coefficient 1 is

$$\left\{\sum_{j=1}^{n+1} \frac{\lambda_j g(z)}{(z-z_j)};\ \lambda_j \geq 0, \sum \lambda_j = 1\right\}.$$

This result follows from Corollary 1 by taking $s = 1$ and $\mathfrak{L}^1 = e_0^n/n!$. (Let e_z^i denote the functional given by $e_z^i f = f^{(i)}(z)$.) For it is evident that Haar Assumptions I and II hold and that $\tilde{e}_j(z) = g(z)/(z - z_j)$, $j = 1, 2, \cdots, n+1$. Further, since $Q \in P_n$ and $Q(z_i) = 0$ $(i = 1, \cdots, n+1)$, therefore $Q \equiv 0$, whence $\beta = A^1 = 1$. Finally $\mathfrak{L}^1 \tilde{e}_j \neq 0$, $j = 1, 2, \cdots, n+1$, and the statement of the example is therefore seen to be just the statement of Corollary 1 in this case.

Example 1 is a special case of the following example.

Example 2 (Simple n-sequences, Shisha-Walsh [5]). Let $E = \{z_s, \cdots, z_{n+1}\}$ and $g(z) = \Pi_{i=s}^{n+1}(z - z_i)$. Let $0 \leq l \leq s$ and $q = n - s + l + 1$ and suppose $0 \notin E$ if $0 < l$. Then the set of infrapolynomials $\Sigma_{i=0}^n c_i z^i$ on E with $c_0, c_1, \cdots, c_{l-1}$ and $c_q, c_{q+1}, \cdots, c_n$ fixed is

$$\left\{Q(z) + \beta z^l \sum_{j=s}^{n+1} \frac{\lambda_j g(z)}{(z-z_j)};\ \lambda_j \geq 0,\ \sum \lambda_j = 1\right\},$$

where

$$Q(z) = Q_1(z) + z^l L(z), \qquad Q_1(z) = \sum_{i=0}^{l-1} c_i z^i + \sum_{i=q+1}^{n} c_i z^i,$$

$L(z)$ is the $(q-l)$-degree Lagrange interpolating polynomial such that $L(z_i) = -Q_1(z_i)/z_i^l$, $i = s, \cdots, n+1$, and $\beta = c_q - Q^{(q)}(0)/q!$.

To see that this result follows from Corollary 1, assume first that $l < s$ and take $\mathfrak{L}^i = e_0^{j_i}/j_i!$, where $j_i = i - 1$, $1 \leq i \leq l$, and $j_i = q - i + s$, $l < i \leq s$. Then $\tilde{e}_j(z) = z^l g(z)/(z - z_j)$, $s \leq j \leq n+1$. Indeed $\tilde{e}_j(z)$ is of degree q with highest coefficient 1 so that $\mathfrak{L}^s \tilde{e}_j = \tilde{e}_j^{(q)}(0)/q! = 1$. Also, since $\tilde{e}_j(z) = \Sigma_{i=l}^q a_i^j z^i$, we have $\mathfrak{L}^i \tilde{e}_j = 0$ $(1 \leq i \leq s-1)$, $s \leq j \leq n+1$. Furthermore it is immediately checked that the condition $0 \notin E$, if $0 < l$, insures that the Haar Assumptions I and II hold. Thus, we again have that the statement of the example is just the statement of Corollary 1 specified to this case.

In the case $l = s$, i.e. $q = n + 1$, apply Corollary 1 to $P_{n+1}(A')$, where $A' = A_1, A_2, \cdots, A_s, 0$. Here $\mathfrak{L}^i = e_0^{i-1}/(i-1)!$, $1 \leq i \leq s$. Define also $\mathfrak{L}^{s+1} = e_0^{n+1}/(n+1)!$. The argument is analogous to that in the case $l < s$. (Note that $c_q = c_{n+1} = 0$.)

Example 2 with $l = s$ is a special case of the following example.

Example 3 (Infrapolynomials with prescribed values at given points, Walsh-Shisha [6]). Let $E = \{z_s, \cdots, z_{n+1}\}$, let $\xi_1, \xi_2, \cdots, \xi_k$ be distinct points of $\mathcal{C}_z$, and for each $j = 1, 2, \cdots, k$, let there be given complex values $w_j^{(0)}, w_j^{(1)}, \cdots, w_j^{(m_j)}$, where $\sum_{j=1}^{k}(m_j + 1) = s$. Let

$$g(z) = \prod_{i=s}^{n+1} (z - z_i) \quad \text{and} \quad h(z) = \prod_{j=1}^{k} (z - z_j)^{m_j+1}.$$

Then the set of infrapolynomials $p(z) = \sum_{i=0}^{n} c_i z^i$ on E with $p^{(\nu)}(\xi_j) = w_j^{(\nu)}$, $\nu = 0, 1, \cdots, m_j$, $j = 1, 2, \cdots, k$, is

$$\left\{ Q(z) + \beta h(z) \sum_{j=s}^{n+1} \frac{\lambda_j q(z)}{(z - z_j)};\ \lambda_j \geq 0, \sum \lambda_j = 1 \right\},$$

where $Q(z)$ is the unique $(n + 1)$st degree polynomial such that $Q^{(\nu)}(\xi_j) = w_j^{(\nu)}$ and $Q(z_i) = 0$, $s \leq i \leq n + 1$, and $\beta = -Q^{(n+1)}(0)/(n + 1)!$.

To see that this result follows from Corollary 1, argue as in Example 2 (where $l = s$), observing that $\tilde{e}_j(z) = h(z)g(z)/(z - z_j)$, $s \leq j \leq n + 1$.

Just as Example 3 follows from Corollary 1 analogously as in Example 2 (with $l = s$), the following example follows from Corollary 1 analogously as in Example 2 (with $l < s$).

Example 4. Consider the situation of Example 3 except that $\sum_{j=1}^{k}(m_j + 1) = t < s$. Then the set of infrapolynomials $p(z) = \sum_{i=0}^{n} c_i z^i$ on E with $p^{(\nu)}(\xi_j) = w_j^{(\nu)}$, $\nu = 0, 1, \cdots, m_j$, $j = 1, 2, \cdots, k$, and with $c_q, \cdots, c_n$ prescribed (where $q = n - s + t + 1$) is

$$\left\{ Q(z) + \beta h(z) \sum_{j=s}^{n+1} \frac{\lambda_j g(z)}{(z - z_j)};\ \lambda_j \geq 0, \sum \lambda_j = 1 \right\},$$

where $Q(z)$ is the unique element of P_n such that $Q^{(\nu)}(\xi_j) = w_j^{(\nu)}$ $(\nu = 0, 1, \cdots, m_j)$, $Q^{(i)}(0)/i! = c_i$ $(q + 1 \leq i \leq n)$, $Q(z_i) = 0$ $(i = s, \cdots, n + 1)$, and $\beta = c_q - Q^{(q)}(0)/q!$.

The following are examples of Theorem 2 in case $r \geq 0$.

Example 5. Let $E = \{z_1, \cdots, z_{n+r+1}\}$ and $g(z) = \prod_{i=1}^{n+r+1}(z - z_i)$. For each $k = 0, 1, \cdots, r$, let σ_j^k be the kth symmetric function ($\sigma_j^0 = 1$) of the $n + r$ numbers in $E - \{z_j\}$, $j = 1, \cdots, n + r + 1$. Then the set of nth degree infrapolynomials on E with leading coefficient 1 is

$$\left\{ \sum_{j=1}^{n+r+1} \frac{\lambda_j g(z)}{(z - z_j)};\quad \lambda \text{ is minimal subject to } \sum \lambda_j = 1 \text{ and } \sum \frac{\lambda_j \sigma_j^k}{\sigma_j^r} = 0 \ (k = 0, 1, \cdots, r - 1) \right\}.$$

This follows immediately from Example 1 and from an analysis of the condition $F\lambda = G$ in this case.

Note 5. Notice that the set of infrapolynomials in Example 5 includes

$$\left\{\sum_{j=1}^{n+r+1} \frac{\lambda_j g(z)}{(z - z_j)};\ \lambda_j \geq 0,\ \sum \lambda_j = 1,\ \sum \frac{\lambda_j \sigma_j^k}{\sigma_j^r} = 0\ \ (k = 0, 1, \cdots, r-1)\right\}.$$

We turn now to some results on the location of zeros of infrapolynomials on E_r with respect to $P_n(A)$. The situation is that of Theorem 2. Let $\{\lambda^*\} = \{\lambda^*;\ \lambda^*$ yields $\min_{F\lambda = G}\|\lambda\|\}$, so that $\{\lambda^*\}$ is a subset of complex $(n + r - s + 2)$-space.

Theorem 3. *In Theorem* 2, *if* $\{\lambda^*\}$ *is bounded, and if* $\deg Q > \max \deg \tilde{e}_j$, $s \leq j \leq n + r + 1$, *then* $\exists H < \infty$ *such that, if* $I(z_0) = 0$ *for some infrapolynomial* I, *then* $|z_0| \leq H$.

Proof. $I(z) = Q(z) + P(z)$, where $Q(z) = a_q z^q + a_{q-1} z^{q-1} + \cdots$, $P(z) = b_p z^p + b_{p-1} z^{p-1} + \cdots$ and $a_q b_p \neq 0$. Since $\{\lambda^*\}$ is bounded, all the coefficients b_j of $P(z)$ are bounded in absolute value by $|\beta|(n + 1)mM$, where $m = \max_{\lambda^*, j} |\lambda_j^*|$ and M is the maximum modulus of all the coefficients of the $\tilde{e}_j$. Thus $0 = I(z_0) = Q(z_0) + P(z_0)$, whence

$$z_0 = a_q^{-1}(-a_{q-1} - a_{q-2} z_0^{-1} - \cdots - a_0 z_0^{-(q-1)} - b_p z_0^{-(q-p-1)} - \cdots - b_0 z_0^{-(q-1)}).$$

Since $q > p \geq 0$, we can set

$$H = \max\left\{1,\ |a_q|^{-1}\left(\sum_0^{q-1} |a_i| + \sum_0^p |b_j|\right)\right\}. \quad \square$$

Corollary 2. *In Corollary* 1, *if* $\mathfrak{L}^s e_j \neq 0$ $(s \leq j \leq n + 1)$ *and if* $\deg Q > \max \deg \tilde{e}_j$, $s \leq j \leq n + 1$, *then* $H < \infty$ *such that, if* $I(z_0) = 0$ *for some infrapolynomial* I, *then* $|z_0| \leq H$.

Proof. Clearly from (*), $|\lambda_j^*| = |\beta|\,|\lambda_j'| \leq |\beta|$, $s \leq j \leq n + 1$, and so $\{\lambda^*\}$ is bounded. The conclusion then follows from Theorem 3. □

Let smp R denote the smallest power of z occurring in R.

Theorem 4. *In Theorem* 2, *if* $\{\lambda^*\}$ *is bounded and if* $\operatorname{smp} Q < \min \operatorname{smp} \tilde{e}_j$, $s \leq j \leq n + r + 1$, *then* $\exists h > 0$ *such that, if* $I(z_0) = 0$ *for some infrapolynomial* I, *then* $|z_0| \geq h$.

Proof. $I(z) = Q(z) + P(z)$, where $Q(z) = a_m z^m + a_{m+1} z^{m+1} + \cdots$ and

$P(z) = b_p z^p + b_{p+1} z^{p+1} + \cdots + b_q z^q$. Again, since $\{\lambda^*\}$ is bounded, all the coefficients b_j are bounded in absolute value. Thus $0 = I(z_0) = Q(z_0) + P(z_0)$, whence

$$1 = a_m^{-1}(-a_{m+1} z_0 - \cdots - a_q z_0^{q-m} - b_p z_0^{p-m} - \cdots - b_q z_0^{q-m}) = t(z_0).$$

Since $q \geq p > m \geq 0$, $\exists h > 0$ such that $|t(z)| \leq ½$ if $|z| \leq h$. □

Note 6. If either $s = 1$ or $A_i = 0$ $(1 \leq i \leq s-1)$, then $Q \equiv 0$ and the hypotheses of Theorem 4 reduce to $\{\lambda^*\}$ being bounded.

Corollary 3. *In Corollary* 1, *if* $\mathfrak{L}^s e_j \neq 0$ $(s \leq j \leq n+1)$ *and if* smp $Q <$ min smp $\tilde{e}_j$, $s \leq j \leq n+1$, *then* $\exists h > 0$ *such that, if* $I(z_0) = 0$ *for some infrapolynomial* I, *then* $|z_0| \geq h$.

Note 7. In the case of Example 2 above (simple n-sequences), Corollaries 2 and 3 of this paper give Theorem 6 of [5].

Remark. Theorems 1 and 2 and Corollary 1 are valid if, for each $m = 0, 1, 2, \cdots$, P_m is replaced by V_m, an arbitrary $(m+1)$-dimensional subspace of $C(\mathcal{C}_z)$ satisfying $V_m \subset V_{m+1}$, and $f^{(q)}(0)$ is replaced by $K_q f$, where $K_1, K_2, \cdots$ is a sequence of linear functionals defined on $\bigcup V_m$ such that $V_m = V_{m+1} \cap$ nullspace of K_{m+1}. Also Theorems 3 and 4 and their corollaries have obvious analogues in this general situation.

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