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COMMUTATORS OF SINGULAR INTEGRALS WITH C^1-KERNELS

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ABSTRACT. The commutator of two Calderón-Zygmund singular integral operators is shown to be smoothing of order one under reduced smoothness assumptions on the kernels.

Introduction. In this note we consider singular integral operators, K, of the form,

$$Kf(x) = \lim_{\epsilon \to 0+} \int_{|x-y|>\epsilon} k(x, x-y) f(y)\, dy,$$

where $k(x, y)$ is defined on $\mathbf{R}^n \times (\mathbf{R}^n \backslash \{0\})$ and satisfies:

(a) $k(x, y) = \lambda^{-n} k(x, y)$, $\lambda > 0$, $y \neq 0$;

(b) $\int_\Sigma k(x, \sigma)\, d\sigma = 0$, where Σ is the unit sphere in $\mathbf{R}^n$.

Under additional mild assumptions on $k(x, y)$ the above limit exists in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, and K is a continuous operator on $L^p(\mathbf{R}^n)$ (see [4]).

It was proved by A. P. Calderón in [3] that if K_1 and K_2 are singular integral operators with corresponding kernels $k_1(x, y)$ and $k_2(x, y)$ such that for each $j, l = 1, \cdots, n$ the functions $k_i(x, y)$, $\partial k_i(x,y)/\partial x_j$, $\partial k_i(x,y)/\partial y_l$, $\partial^2 k_i(x, y)/\partial x_i \partial y_l$, $\partial^2 k_i(x, y)/\partial y_l^2$ belong to $L^\infty(\mathbf{R}^n \times \Sigma)$ $(i = 1, 2)$, then the commutator, $K_1K_2 - K_2K_1$, is a smoothing operator. In fact, the operators

$$(K_1K_2 - K_2K_1)\frac{\partial}{\partial x_j} \quad \text{and} \quad \frac{\partial}{\partial x_j}(K_1K_2 - K_2K_1)$$

are bounded on $L^p(\mathbf{R}^n)$. The way to prove this result is to study the continuity in $L^p(\mathbf{R}^n)$ of the operators,

$$(1) \qquad (K_1K_2 - K_1 \circ K_2)\frac{\partial}{\partial x_j} \quad \text{and} \quad \frac{\partial}{\partial x_j}(K_1K_2 - K_1 \circ K_2),$$

where $K_1 \circ K_2$ is the pseudoproduct of K_1 and K_2. More specifically, if we define

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$$\sigma(K_i)(x,\xi) = \lim_{\epsilon\to 0} \frac{1}{(2\pi)^n} \int_{\epsilon<|y|<1/\epsilon} k_i(x,y) e^{i\langle \xi, y\rangle}\, dy,$$

then $K_1 \circ K_2$ is the unique singular integral operator such that

$$\sigma(K_1 \circ K_2)(x,\xi) = \sigma(K_1)(x,\xi)\,\sigma(K_2)(x,\xi).$$

In this work our main objective is to show that Calderón's result still holds with no assumptions on the existence of the second order derivative (Theorem 2).

Lemma 1. (i) *Assume* $f(x,\sigma)$ *is defined on* $\mathbf{R}^n \times \Sigma$ *and belongs to* $L^p(\mathbf{R}^n \times \Sigma)$, $1 < p < \infty$.

(ii) *Let* $h(x,y)$ *denote a function defined on* $\mathbf{R}^n \times (\mathbf{R}^n \backslash \{0\})$, *homogeneous of degree* $-n-1$ *in* y, *even in the variable* y, *and such that* $\|(\|h(x,\cdot)\|_{L^q(\Sigma)})\|_{L^\infty(\mathbf{R}^n)} < \infty$, *where* $1/p + 1/q = 1$.

(iii) *Assume* $a(x,\sigma)$ *is a measurable function on* $\mathbf{R}^n \times \Sigma$ *such that for each* σ, $a(\cdot,\sigma) \in C^1(\mathbf{R}^n)$ *with* $\sup_x |\nabla_x a(x,\sigma)| \in L^q(\Sigma)$, $1/p + 1/q = 1$. *Set*

$$\tilde{C}f(x) = \int_{\Sigma} \lim_{\epsilon\to 0+} \left(\int_{|x-y|>\epsilon} h(x, x-y)[a(x,\sigma) - a(y,\sigma)] f(y,\sigma)\, dy \right) d\sigma,$$

where the limit is understood in $L^p(\mathbf{R}^n)$. *Then, if* $1/p + 1/q = 1$,

$$\|\tilde{C}f\|_{L^p(\mathbf{R}^n)} \le C_p \|(\|h(x,\cdot)\|_{L^q(\Sigma)})\|_{L^\infty(\mathbf{R}^n)} \cdot \left\| \sup_x |\nabla_x a(x,\sigma)| \right\|_{L^q(\Sigma)} \|f\|_{L^p(\mathbf{R}^n \times \Sigma)}.$$

Proof. Set

$$\tilde{C}_\sigma f(x) = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} h(x, x-y)[a(x,\sigma) - a(y,\sigma)] f(y,\sigma)\, dy.$$

By Minkowski's inequality,

$$\|\tilde{C}f\|_{L^p(\mathbf{R}^n)} \le \int_{\Sigma} \|\tilde{C}_\sigma f\|_{L^p(\mathbf{R}^n)}\, d\sigma.$$

Since h is even and homogeneous of degree $-n-1$ in y,

$$\tilde{C}_\sigma(f)(x) = \frac{1}{2} \int_{\Sigma} h(x,\tau) H_\sigma(f)(x,\tau)\, d\tau,$$

where

$$H_\sigma(f)(x,\tau) = \lim_{\epsilon\to 0+} \int_{|s|>\epsilon} \frac{[a(x,\sigma) - a(x - s\tau, \sigma)]}{s^2} f(x - s\tau, \sigma)\, ds.$$

Hence

$$\int_{\mathbf{R}^n} |\tilde{C}_\sigma f(x)|^p\,dx \le C \|(\|b(x,\cdot)\|_{L^q(\Sigma)})\|^p_{L^\infty(\mathbf{R}^n)} \int_\Sigma \int_{\mathbf{R}^n} |H_\sigma(f)(x,\tau)|^p\,dx\,d\tau.$$

For each $\tau \in \Sigma$, write $x = r\tau + \omega_\tau$, where $\langle \omega_\tau, \tau\rangle = 0$. Then, from Calderón's 1-dimensional result in [2] we have

$$\int_{\mathbf{R}^n} |H_\sigma f(x,\tau)|^p\,dx \le C\left(\sup_x |\nabla_x a(x,\sigma)|\right)^p \int_{\mathbf{R}^n} |f(x,\sigma)|^p\,dx.$$

Hence,

$$\|\tilde{C}f\|_{L^p(\mathbf{R}^n)} \le C_p \|(\|b(x,\cdot)\|_{L^q(\Sigma)})\|_{L^\infty(\mathbf{R}^n)} \cdot \int_\Sigma \sup_x |\nabla_x a(x,\sigma)|\, \|f(\cdot,\sigma)\|_{L^p(\mathbf{R}^n)}\,d\sigma.$$

An application of Hölder's inequality concludes the proof.

Theorem 1. (i) *For each $x \in \mathbf{R}^n$ let $k(x, y)$ be homogeneous of degree $-n$. Assume $k(x, y)$ and $\partial k(x,y)/\partial y_j \in L^q_{\mathrm{loc}}(\mathbf{R}^n\setminus\{0\})$, $1 < q < \infty$, with*

$$\left(\|k(x,\cdot)\|_{L^q(\Sigma)} + \left\|\frac{\partial}{\partial y_j} k(x,\cdot)\right\|_{L^q(\Sigma)}\right) \in L^\infty(\mathbf{R}^n).$$

(ii) *Assume $a(x,\sigma)$ is defined and measurable over $\mathbf{R}^n \times \Sigma$ with $\|\sup_x |\nabla_x a(x,\sigma)|\,\|_{L^q(\Sigma)} < \infty$.*

(iii) *Assume $f(x,\sigma)$ is defined over $\mathbf{R}^n \times \Sigma$ and belongs to $L^p(\mathbf{R}^n \times \Sigma)$, $1 < p < \infty$. Set*

$$C(f)(x) = \int_\Sigma \lim_{\epsilon\to 0+} \left[\int_{|x-y|>\epsilon} (a(x,\sigma) - a(y,\sigma)) \frac{\partial}{\partial y_i} k(x, x-y) f(y,\sigma)\,dy\right] d\sigma,$$

where the limit is understood in $L^p(\mathbf{R}^n)$. If $1/p + 1/q = 1$, then

$$\|Cf\|_{L^p(\mathbf{R}^n)} \le C_p \left\|\left(\|k(x,\cdot)\|_{L^q(\Sigma)} + \left\|\frac{\partial}{\partial y_j} k(x,\cdot)\right\|_{L^q(\Sigma)}\right)\right\|_{L^\infty(\mathbf{R}^n)} \cdot \left\|\sup_x |\nabla_x a(x,\sigma)|\right\|_{L^q(\Sigma)} \|f\|_{L^p(\mathbf{R}^n\times\Sigma)}.$$

Proof. By writing

$$k(x,y) = \frac{k(x,y) + k(x,-y)}{2} + \frac{k(x,y) - k(x,-y)}{2},$$

Lemma 1 takes care of the odd part of k; therefore we may assume $k(x, y)$ is even in y.

Let R_i denote the ith Riesz transform, i.e. $R_i f$ is defined by the formula, $R_i f = F^{-1}(x_i F(f)/|x|)$. ($F$ and F^{-1} denote, respectively, the Fourier and inverse Fourier transforms.) Then $\Sigma_{i=1}^{n} R_i^2$ = identity. Set

$$K_{\partial_j} f(x) = \lim_{\epsilon \to 0+} \int_{|y|>\epsilon} \frac{\partial}{\partial y_j} k(x, y) f(x-y)\, dy$$

and

$$Kf(x) = \lim_{\epsilon \to 0+} \int_{|y|>\epsilon} k(x, y) f(x-y)\, dy.$$

Exactly as was shown in [1] we can write

$$K_{\partial_j} R_i f(x) = \lim_{\epsilon \to 0+} \int_{|y|>\epsilon} h_{i,j}(x, y) f(x-y)\, dy,$$

where $h_{i,j}(x, y)$ satisfies the condition of Lemma 1 and

$$\|h_{i,j}(x, \cdot)\|_{L^q(\Sigma)} \leq C_q \left\| \frac{\partial}{\partial y_j} k(x, \cdot) \right\|_{L^q(\Sigma)}.$$

For fixed $\sigma \in \Sigma$, let a_σ and f_σ denote the functions $a(\cdot, \sigma)$ and $f(\cdot, \sigma)$.

$$Cf = \sum_{i=1}^{n} \int_{\Sigma} (a_\sigma K_{\partial_j} - K_{\partial_j} a_\sigma) R_i (R_i f_\sigma)\, d\sigma.$$

$$Cf = \sum_{i=1}^{n} \int_{\Sigma} [a_\sigma (K_{\partial_j} R_i) - (K_{\partial_j} R_i) a_\sigma] R_i (f_\sigma)\, d\sigma$$

$$+ \sum_{i=1}^{n} K_{\partial_j} \int_{\Sigma} (R_i a_\sigma - a_\sigma R_i) R_i (f_\sigma)\, d\sigma.$$

Using Lemma 1 we see that as an operator on $L^p(\mathbf{R}^n)$, the norm of each term in the first summation is dominated by

$$C_p \left\| \left(\left\| \frac{\partial}{\partial y_j} k(x, \cdot) \right\|_{L^q(\Sigma)} \right) \right\|_{L^\infty(\mathbf{R}^n)} \left\| \sup_x |\nabla_x a_\sigma(x)| \right\|_{L^q(\Sigma)} \|f\|_{L^p(\mathbf{R}^n \times \Sigma)}.$$

For the second summation we first observe that $K_{\partial_j} = K\partial_j$ where ∂_j denotes the operation of taking the jth partial derivative. Hence

$$K_{\partial_j} \left(\int_{\Sigma} (R_i a_\sigma - a_\sigma R_i) R_i (f_\sigma) \right)$$

$$= K \left(\int_{\Sigma} [(\partial_j R_i) a_\sigma - a_\sigma (\partial_j R_i)] R_i (f_\sigma)\, d\sigma \right) - K \int_{\Sigma} (\partial_j a_\sigma) R_i^2 (f_\sigma).$$

Again, by Lemma 1,

$$\left\| \int_{\Sigma} [(\partial_j R_i) a_\sigma - a_\sigma (\partial_j R_i)] R_i (f_\sigma)\, d\sigma \right\|_{L^p(\mathbf{R}^n)}$$

$$\leq C_p \left\| \sup_x |\nabla_x a_\sigma(x)| \right\|_{L^q(\Sigma)} \|f\|_{L^p(\mathbf{R}^n \times \Sigma)}.$$

The same estimate is valid for the norm of the function, $\int_\Sigma (\partial_j a_\sigma) R_i^2(f_\sigma)\, d\sigma$. Since the norm of the operator K is bounded by $C_p \|(\|k(x,\cdot)\|_{L^q(\Sigma)})\|_{L^\infty(\mathbf{R}^n)}$ (see [4, Theorem 2]), this together with the above estimates gives the desired result.

Theorem 2. *Let K_1 and K_2 be singular integral operators of the Calderón-Zygmund type with corresponding kernels $k_1(x, y)$ and $k_2(x, y)$. Assume for each $i = 1, 2$, $j = 1, \cdots, n$, $k_i(x, y)$, $\partial k_i(x, y)/\partial y_j$, $\partial k_i(x, y)/\partial x_j \in C(\mathbf{R}^n \times \Sigma)$, and, in addition, assume $\partial k_i(x, y)/\partial x_j$ is Hölder continuous in y on Σ with a Hölder constant and exponent uniform for $x \in \mathbf{R}^n$.*

For each p, $1 < p < \infty$, and each $j = 1, \cdots, n$ the operators of (1) *are bounded operators on $L^p(\mathbf{R}^n)$.*

Proof. We will prove the result for the operator $(K_1K_2 - K_1 \circ K_2)(\partial/\partial x_j)$. The proof for the second operator follows almost immediately from this result.

First assume K_2 is given by an odd kernel, $k_2(x, y)$. Let $k_{2,\sigma}(x)$ denote the function $k_2(x, \sigma)$, where $\sigma \in \Sigma$. Again, for σ fixed, let $k_{2,\sigma}$ denote the operation of multiplication by $k_{2,\sigma}(x)$. Then $K_2 = \int_\Sigma k_{2,\sigma} H_\sigma\, d_\sigma$, where

$$H_\sigma(f)(x) = \lim_{\epsilon \to 0+} \frac{1}{2} \int_{|s| > \epsilon} \frac{f(x - s\sigma)}{s}\, ds.$$

$$K_1K_2 = \int_\Sigma K_1(k_{2,\sigma} H_\sigma)\, d\sigma, \qquad K_1 \circ K_2 = \int_\Sigma k_{2,\sigma} K_1 H_\sigma\, d\sigma.$$

Hence

$$(K_1K_2 - K_1 \circ K_2)\partial_j = \int_\Sigma (K_1 k_{2,\sigma} - k_{2,\sigma} K_1)\partial_j H_\sigma\, d\sigma$$

$$= \int_\Sigma [(K_1\partial_j)k_{2,\sigma} - k_{2,\sigma}(K_1\partial_j)] H_\sigma\, d\sigma - K_1 \int_\Sigma (\partial_j k_{2,\sigma}) H_\sigma\, d\sigma.$$

Hence, as an operator on $L^p(\mathbf{R}^n)$,

$$\|(K_1K_2 - K_1 \circ K_2)\partial_j\| \le C_p \sup_x \left(\left\| \frac{\partial}{\partial y_j} k_1(x, \cdot) \right\|_{L^q(\Sigma)} + \|k_1(x, \cdot)\|_{L^q(\Sigma)} \right) \cdot \left\| \sup_x |\nabla_x k_2(x, \cdot)| \right\|_{L^q(\Sigma)}.$$

In the case K_2 is even then

$$(K_1K_2 - K_1 \circ K_2) = \sum_i [(K_1K_2 - K_1 \circ K_2)R_i]R_i.$$

We observe that $K_1K_2R_i = K_1(K_2 \circ R_i)$ and that $(K_1 \circ K_2)R_i = K_1 \circ (K_2 \circ R_i)$. The operator $K_2 \circ R_i$ is given by an odd kernel $k_{2,i}(x, y)$ with $\sup_{x,\sigma} |\nabla_x k_{2,i}(x, \sigma)| < \infty$. In fact, if $k_2(x, \sigma)$ is Hölder continuous of order δ in σ, then this last supremum is bounded by

$$C_\delta \left[\frac{\sum_{j=1}^n \sup_x \sup_{\sigma_1,\sigma_2 \in \Sigma} (|\partial k_2(x, \sigma_1)/\partial x_j - \partial k_2(x, \sigma_2)/\partial x_j|)}{|\sigma_1 - \sigma_2|^\delta} + \sum_{j=1}^n \sup_x \sup_{\sigma \in \Sigma} \left| \frac{\partial}{\partial x_j} k_2(x, \sigma) \right| \right].$$

Applying our estimate in the odd case to $[K_1(K_2 \circ R_i) - K_1 \circ (K_2 \circ R_i)](\partial/\partial x_j)$, we obtain our desired result.

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