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AN EXTENSION OF THE ERDÖS-RÉNYI NEW LAW OF LARGE NUMBERS

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ABSTRACT. If S_n is the nth partial sum of a sequence of independent, identically distributed random variables $X_1, X_2, \cdots$ such that $E(X_1) = 0$ and $E(\exp(tX_1)) < \infty$ for some nonempty interval of t's, then, for a wide range of positive numbers λ, Erdös and Rényi (1970) showed that $\Sigma(N, [C(\lambda)\log N])$ converges with probability one to λ as $N \to \infty$, where $\Sigma(N, K)$ is the maximum of the $N - K + 1$ averages of the form $K^{-1}(S_{n+K} - S_n)$ for $0 \le n \le N - K$, and $C(\lambda)$ is a known constant depending on λ and the distribution of X_1. The objective of the present article is to state and prove the Erdös-Rényi theorem for the $N - K + 1$ "averages" of the form $K^{-1/r}(S_{n+K} - S_n)$, where $1 < r < 2$. This form of the Erdös-Rényi theorem arises from the extended form of the strong law of large numbers which asserts that, if $E(|X_1|^r) < \infty$ for some r, $1 \le r < 2$, and $E(X_1) = 0$, then $n^{-1/r}S_n$ converges with probability one to 0 as $n \to \infty$.

0. Introduction. If $\{X_n : 1 \le n < \infty\}$ is a sequence of independent, identically distributed (i.i.d.) random variables, with partial sums $S_n = \Sigma_{k=1}^n X_k$, then there are $N - K + 1$ successive averages of the form $K^{-1}(S_{n+K} - S_n)$, one for each value of n between 0 and $N - K$, inclusive. We denote the largest of these averages by $\Sigma(N, K)$, i.e.

$$\Sigma(N, K) = \max\{K^{-1}(S_{n+K} - S_n): 0 \le n \le N - K\}.$$

Then, by the Borel-Cantelli lemma, $P\{\lim_{N\to\infty}\Sigma(N, 1) = \mu\} = 1$ if $\mu \le \infty$ is the essential supremum of X_1. If $E(X_1) = 0$, the strong law of large numbers asserts that $P\{\lim_{N\to\infty}\Sigma(N, N) = 0\} = 1$. What happens to $\Sigma(N, K)$ for values of K between 1 and N was the subject of the 1970 article [4] by Erdös and Rényi. If X_1 has moment-generating function $\phi(t) = E(\exp(tX_1)) < \infty$ for a sufficiently large interval of t's, then for every λ between 0 and μ (exclusive

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of the endpoints), Erdös and Rényi showed that there exists a constant $C(\lambda)$, whose value can be determined, depending on λ and the distribution of X_1, such that

$$P\left\{\lim_{N\to\infty} \Sigma(N, [C(\lambda)\log N]) = \lambda\right\} = 1.$$

Here $[y]$ denotes the greatest integer $\leq y$. Erdös and Rényi called their result "a new law of large numbers" because of the fundamental relation it bears to the strong law of large numbers. The theorem in [2] extends the Erdös-Rényi result to the case of weighted sums of i.i.d. random variables.

It is the objective of the present article to extend the Erdös-Rényi theorem to averages of the form $K^{-1/r}(S_{n+K} - S_n)$ for $1 < r < 2$. Toward this end, we define

$$\Sigma_r(N, K) = \max\{K^{-1/r}(S_{n+K} - S_n): 0 \leq n \leq N - K\}.$$

Again $P\{\lim_{N\to\infty}\Sigma_r(N, 1) = \mu\} = 1$ and, if $E(X_1) = 0$ and $E(|X_1|^r) < \infty$, the strong law of large numbers asserts that $P\{\lim_{N\to\infty}\Sigma_r(N, N) = 0\} = 1$. The strong law of large numbers in the form $P(n^{-1/r}S_n \to 0) = 1$ for $1 < r < 2$ can be found in Loève's text [7, p. 243]. In the present article, we intend to show that, for every positive λ, $0 < \lambda < \infty$,

$$P\left\{\lim_{N\to\infty} \Sigma_r(N, [(2\lambda^{-2}\log N)^{r/(2-r)}]) = \lambda\right\} = 1$$

as long as there exists some nondegenerate interval of t's, containing the origin in its interior, for which $\phi(t) = E(\exp(tX_1)) < \infty$. Note that, in the present situation, $\phi(t)$ is required to exist only in a nondegenerate interval, not necessarily a sufficiently large one, and that the constant $2\lambda^{-2}$ depends only on λ and is independent of the distribution of X_1.

Erdös and Rényi based their proof on the large deviation theorem of Cramér [3], in the form due to Bahadur and Ranga Rao [1], which asserts that

$$P(S_n \geq \lambda n) \sim (2\pi n)^{-1/2}\rho^n b_n,$$

where $\rho = \exp(-1/C(\lambda))$, $\{b_n: 1 \leq n < \infty\}$ is a bounded sequence, and the symbol "$\sim$" indicates that the ratio of the two sides tends to 1 as $n \to \infty$. To provide a point of departure for the theorem of the present article, we need a large deviation theorem also, one due to Petrov [8], which can be found in the most easily usable form in the monograph of Ibragimov and Linnik [6]. Petrov's theorem, which is a generalization of and has the same form as the original Cramér theorem, implies in our situation, for $1 < r < 2$, that

$$P(S_n \geq \lambda n^{1/r}) \sim (2\pi n^{\alpha})^{-1/2}\lambda^{-1}\exp\{-\tfrac{1}{2}\lambda^2 n^{\alpha}(1 + o(1))\}$$

as $n \to \infty$, where $\alpha = (2 - r)/r$. Note that if $r = 1$, then $\alpha = 1$, and the above statement is somewhat reminiscent of the Bahadur-Ranga Rao result.

Petrov's large deviation theorem and the corollary of it that we use are recorded in §1. §2 contains the extension of the Erdös-Rényi theorem. In §3, we briefly compare the main theorem of this article with the original Erdös-Rényi theorem, focusing on the relationship between $C(\lambda)$ and $2\lambda^{-2}$.

1. **Petrov's theorem and its corollary.** We consider throughout the paper a sequence of nondegenerate i.i.d. random variables $\{X_n: 1 \le n < \infty\}$ such that $E(X_1) = 0$, $\mathrm{Var}(X_1) = 1$, and $\phi(t) = E(\exp(tX_1)) < \infty$ for $|t| < B$, where $0 < B \le \infty$.

For the theorem of the present article, we need no condition on the moment-generating function (m.g.f.) ϕ beyond its existence in a nondegenerate interval containing the origin in its interior. Bahadur and Ranga Rao, and therefore Erdös and Rényi, had to require that the function $Q(t) = \phi'(t)/\phi(t)$ take on the value λ at some point $t = t_\lambda$. (An analogous condition is required in the case of weighted sums that is studied in [2].) That assumption is vital in their situation, however, because $C(\lambda)$ turns out to be the reciprocal of $\lambda t_\lambda - \log \phi(t_\lambda)$. Moreover, a sufficient condition for the existence of $t_\lambda = Q^{-1}(\lambda)$ is that $\phi(t) < \infty$ for all real t and $P(X_1 > \lambda) > 0$. Now, if $P(X_1 > \lambda) = 0$, particularly if $\lambda > \mu$, the essential supremum of X_1, then $\Sigma(N, K) \le \mu < \lambda$ for all N and K so that $\lim_{N\to\infty} \Sigma(N, K) = \lambda$ is not possible. In our case, on the other hand, $\Sigma_r(N, K) > \lambda$ is possible even if $\lambda > \mu$, so no such condition relating ϕ and λ is necessary.

We consider a sequence of numbers $\{z_n: 1 \le n < \infty\}$ such that $z_n \to \infty$ and $n^{-1/2} z_n \to 0$ as $n \to \infty$, and we want to study the probability $P(n^{-1/2} S_n \ge z_n)$, where $S_n = \Sigma_{k=1}^{n} X_k$. We will be especially interested in the case $z_n = \lambda n^{\alpha/2}$, where $\alpha = (2 - r)/r$ for $1 < r < 2$. In the case $r = 2$, $\alpha = 2$ and we know by the most elementary form of the central limit theorem that

$$P(n^{-1/2} S_n \ge \lambda) \to 1 - \Phi(\lambda) = (2\pi)^{-1/2} \int_\lambda^\infty \exp(-u^2/2)\, du \quad \text{as } n \to \infty.$$

For $r < 2$, $\alpha > 0$ so that $P(n^{-1/2} S_n \ge \lambda n^{\alpha/2})$ tends to 0 as $n \to \infty$. The rate at which this probability tends to 0 when $r = 1$ is the subject of Cramér [3] and Bahadur and Ranga Rao [1]. We now deal with the case $1 < r < 2$.

Under the conditions discussed above, we have the following large deviation theorem, which can be found in [6, p. 171]:

Theorem 1.1 (Petrov). *If $\phi(t) < \infty$ for $|t| < B$, where $0 < B \le \infty$, and $z_n \to \infty$ and $n^{-1/2} z_n \to 0$ as $n \to \infty$, then*

$$P(n^{-\frac{1}{2}} S_n \geq z_n) = \{1 - \Phi(z_n)\} \exp\{n^{-\frac{1}{2}} z_n^3 \lambda(n^{-\frac{1}{2}} z_n)\}\{1 + O(n^{-\frac{1}{2}} z_n)\},$$

where

$$\Phi(z) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{z} \exp(-t^2/2)\,dt$$

and $\lambda(z)$ *is the Cramér series.*

In Petrov's theorem, the Cramér series $\lambda(z) = \sum_{k=0}^{\infty} \lambda_k z^k$ is a power series whose coefficients depend on the moments and semi-invariants of X_1 and which converges for all sufficiently small values of z, the radius of convergence depending on ϕ. Further details about the series are available, for example, in Cramér's original article [3] and in Chapter 7 of Ibragimov and Linnik's book [6]. In addition, by a fact proved in Feller's text [5, p. 175], $z_n \longrightarrow \infty$ implies that

$$(*) \qquad 1 - \Phi(z_n) = (2\pi)^{-\frac{1}{2}} z_n^{-1} \exp(-z_n^2/2)\{1 + o(1)\}$$

as $n \longrightarrow \infty$. Petrov's theorem then yields

Corollary 1.2. *For* $1 < r < 2$ *and* $\alpha = (2-r)/r$, *as* $n \longrightarrow \infty$,

$$P(S_n \geq \beta n^{1/r}) = (2\pi\beta^2 n^{\alpha})^{-\frac{1}{2}}$$
$$\cdot \exp\{-\tfrac{1}{2}\beta^2 n^{\alpha}(1 - \beta n^{(\alpha-1)/2}\lambda(\beta n^{(\alpha-1)/2}))\}(1 + o(1)).$$

Proof. In Theorem 1.1 and (*) above, we take $z_n = \beta n^{(2-r)/2r} = \beta n^{\alpha/2}$, where $0 < \alpha < 1$. Therefore $z_n^{-1} = \beta^{-1} n^{-\alpha/2}$, $z_n^2 = \beta^2 n^{\alpha}$, $z_n^3 = \beta^3 n^{3\alpha/2}$, and $n^{-\frac{1}{2}} z_n = \beta n^{(\alpha-1)/2}$ 0 as $n \longrightarrow \infty$. We can therefore write that

$$P(S_n \geq \beta n^{1/r}) = P(n^{-\frac{1}{2}} S_n \geq \beta n^{\alpha/2}).$$

The result follows by inserting (*) into the conclusion of Theorem 1.1 and then making the substitutions for z_n.

The following corollary expresses the large deviation result in the form needed for the proof of the main theorem:

Corollary 1.3. *If* $\phi(t) < \infty$ *for* $|t| < B$, *where* $0 < B \leq \infty$, *then for all* $\beta > 0$ *and all sufficiently large* n, *there exist numbers* θ_n *depending on* λ *and* ϕ *such that* $\theta_n \longrightarrow 0$ *as* $n \longrightarrow \infty$ *and*

$$\exp\{-\tfrac{1}{2}\beta^2 n^{\alpha}(1 + |\theta_n|)\} \leq 2(2\pi\beta^2 n^{\alpha})^{\frac{1}{2}} P(n^{-1/r} S_n \geq \beta)$$
$$\leq 3 \exp\{-\tfrac{1}{2}\beta^2 n^{\alpha}(1 - |\theta_n|)\},$$

where $1 < r < 2$ *and* $\alpha = (2-r)/r$.

Proof. We apply Corollary 1.2, noting that $0 < \alpha < 1$ and that $\lambda(z)$ converges for all sufficiently small z. We then set

$$\theta_n = \beta n^{(\alpha-1)/2}\lambda(\beta n^{(\alpha-1)/2}),$$

and we take n large enough so that $0.5 < 1 + o(1) < 1.5$.

2. **The law of large numbers.** This section contains the proof of the main theorem of this article, the extension of the Erdös-Rényi new law of large numbers. For $1 < r < 2$ and $S_n = \sum_{k=1}^{n} X_k$, we define

$$\Sigma_r(N, K) = \max\{K^{-1/r}(S_{n+K} - S_n): 0 \leq n \leq N - K\}.$$

We now state and prove the main theorem:

Theorem 2.1. *If* $\{X_n : 1 \leq n < \infty\}$ *is a sequence of i.i.d. random variables with* $E(X_1) = 0$, $\mathrm{Var}(X_1) = 1$, *and m.g.f.* $\phi(t) < \infty$ *for* $|t| < B$, *where* $0 < B \leq \infty$, *and* $1 < r < 2$, *then, for every* $\lambda > 0$,

$$P\left\{\lim_{N\to\infty} \Sigma_r(N, [(2\lambda^{-2}\log N)^{1/\alpha}]) = \lambda\right\} = 1,$$

where $\alpha = (2 - r)/r$.

Proof. We take $K_N = [(2\lambda^{-2}\log N)^{1/\alpha}]$, and to simplify the notation, we set

$$\Sigma(r, \lambda, N) = \Sigma_r(N, K_N) \quad \text{and} \quad S_N(n, \lambda) = K_N^{-1/r}(S_{n+K_N} - S_n).$$

For $\epsilon > 0$ arbitrary, we define $\lambda'' = \lambda + \epsilon$, and we obtain from Corollary 1.3 that

$$\begin{aligned} P(\Sigma(r, \lambda, N) \geq \lambda'') &= P\left(\max_{0 \leq n \leq N-K_N} S_N(n, \lambda) \geq \lambda''\right) \\ &\leq \sum_{n=0}^{N-K_N} P(S_N(n, \lambda) \geq \lambda'') \\ &\leq (N - K_N + 1)3(2\lambda'')^{-1}(2\pi K_N^{\alpha})^{-1/2}\exp\{-\tfrac{1}{2}(\lambda'')^2 K_N^{\alpha}(1 - |\theta''_{K_N}|)\}, \end{aligned}$$

where $\theta''_n = \theta_n(\lambda'')$. We can choose N'' so large that, for $N \geq N''$,

$$\begin{aligned} \tfrac{1}{2}(\lambda'')^2 K_N^{\alpha}(1 - |\theta''_{K_N}|) &= \tfrac{1}{2}\lambda^2(1 + \lambda^{-1}\epsilon)^2\{K_N(K_N + 1)^{-1}\}^{\alpha}(K_N + 1)^{\alpha}(1 - |\theta''_{K_N}|) \\ &\geq \tfrac{1}{2}\lambda^2(1 + \delta_1)2\lambda^{-2}\log N = (1 + \delta_1)\log N \end{aligned}$$

as $K_N + 1 > (2\lambda^{-2}\log N)^{1/\alpha}$, where $\delta_1 > 0$ is a constant depending only on λ and ϵ. Increasing N'', if necessary, so that also $\{K_N(K_N+1)^{-1}\}^\alpha > \frac{1}{2}$ for $N \geq N''$, we have that

$$P(\Sigma(r, \lambda, N) \geq \lambda'') \leq 3N(2\lambda)^{-1}(4\pi\lambda^{-2}\log N)^{-\frac{1}{2}}\exp\{-(1+\delta_1)\log N\}$$
$$\leq 3(\log N)^{-\frac{1}{2}}N^{-\delta_1} \leq 3N^{-\delta_1}.$$

For all large values of N such that $[(2\lambda^{-2}\log N)^{1/\alpha}] = j$, we have $j \leq (2\lambda^{-2}\log N)^{1/\alpha} < j+1$, so that

$$\exp\{\tfrac{1}{2}\lambda^2 j^\alpha\} \leq N < \exp\{\tfrac{1}{2}\lambda^2(j+1)^\alpha\}.$$

We denote $[\exp(\frac{1}{2}\lambda^2 j^\alpha)]$ by E_j, and then we define N_j to be the largest integer such that $[(2\lambda^{-2}\log N)^{1/\alpha}] = j$, so that $N_j \geq E_j$. It follows that, for $j'' = [(2\lambda^{-2}\log N'')^{1/\alpha}]$,

$$\sum_{j=j''}^{\infty} P(\Sigma_r(N_j, j) \geq \lambda'') \leq \sum_{j=j''}^{\infty} 3N_j^{-\delta_1} \leq 3\sum_{j=j''}^{\infty} E_j^{-\delta_1} < \infty,$$

since the series converges by the integral test. Therefore, by the Borel-Cantelli lemma, the probability is one that $\Sigma_r(N_j, j) < \lambda''$ for all but finitely many values of j. But, for each N, such that $N_{j-1} < N \leq N_j$, we have $\Sigma_r(N, j) \leq \Sigma_r(N_j, j)$ because $N \leq N_j$, so that $\Sigma_r(N, j) \geq \lambda''$ implies $\Sigma_r(N_j, j) \geq \lambda''$ as well. Therefore the probability is one that $\Sigma_r(N, j) < \lambda''$ for all but finitely many values of j, where $j = K_N = [(2\lambda^{-2}\log N)^{1/\alpha}]$. It follows that

$$P\left\{\limsup_{N\to\infty} \Sigma_r(N, [(2\lambda^{-2}\log N)^{1/\alpha}]) < \lambda + \epsilon\right\} = 1,$$

and, since $\epsilon > 0$ is arbitrary, we have

$$P\left\{\limsup_{N\to\infty} \Sigma_r(N, [(2\lambda^{-2}\log N)^{1/\alpha}]) \leq \lambda\right\} = 1.$$

On the other hand, we take $\epsilon > 0$ arbitrary and define $\lambda' = \lambda - \epsilon$. Then, using $S_N[m, \lambda]$ to denote $K_N^{-1/r}(S_{(m+1)K_N} - S_{mK_N})$ for integer values of m, we have that

$$P\{\Sigma(r, \lambda, N) < \lambda'\} = P\left\{\max_{0 \leq n \leq N-K_N} S_N(n, \lambda) < \lambda'\right\}$$
$$\leq P\{S_N[m, \lambda] < \lambda' \text{ for } 0 \leq m \leq [NK_N^{-1}] - 1\}$$

since $mK_N = n$ and $0 \leq n \leq N - K_N$ imply that $0 \leq mK_N \leq N - K_N$, and $(m+1)K_N = N$ if and only if $m = NK_N^{-1} - 1$. By independence of the increments, we have

$$P\{\Sigma(r,\lambda,N)<\lambda'\}\le\prod_{m=0}^{[NK_N^{-1}]-1}P\{S_N[m,\lambda]<\lambda'\}\le\{P(K_N^{-1/r}S_{K_N}<\lambda')\}^{[NK_N^{-1}]}.$$

From Corollary 1.3, we can write that

$$P\{K_N^{-1/r}S_{K_N}<\lambda'\}=1-P\{S_{K_N}\ge\lambda'K_N^{1/r}\}$$
$$\le 1-(2\lambda')^{-1}(2\pi K_N^{\alpha})^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}(\lambda')^2K_N^{\alpha}(1+|\theta'_{K_N}|)\},$$

where $\theta'_n=\theta_n(\lambda')$. Now $\lambda'=\lambda-\epsilon$ and $K_N\le(2\lambda^{-2}\log N)^{1/\alpha}$ imply that

$$\tfrac{1}{2}(\lambda')^2K_N^{\alpha}(1+|\theta'_{K_N}|)\le\tfrac{1}{2}\lambda^2(1-\epsilon\lambda^{-1})^2(2\lambda^{-2}\log N)(1+|\theta'_{K_N}|)$$
$$\le(1-\epsilon\lambda^{-1})(\log N)(1+|\theta'_{K_N}|).$$

We can choose N' so large that $(1-\epsilon\lambda^{-1})(1+|\theta'_{K_N}|)<1-3\delta_2$, where $\delta_2>0$ is a constant depending only on λ and ϵ. Therefore

$$P\{K_N^{-1/r}S_{K_N}<\lambda'\}\le 1-(2\lambda)^{-1}(2\pi 2\lambda^{-2}\log N)^{-\frac{1}{2}}N^{-(1-3\delta_2)}$$
$$\le 1-(16\pi\log N)^{-\frac{1}{2}}N^{-(1-3\delta_2)}$$
$$\le 1-N^{-(1-2\delta_2)}\le\exp\{-N^{-(1-2\delta_2)}\},$$

taking N' also large enough so that $(16\pi\log N)^{-\frac{1}{2}}\ge N^{-\delta_2}$. It follows that, for all $N\ge N'$,

$$P\{\Sigma(r,\lambda,N)<\lambda'\}\le\exp\{-[NK_N^{-1}]N^{-(1-2\delta_2)}\}\le\exp\{-N^{\delta_2}\},$$

if we take N' large enough so that $K_N\le N^{\delta_2}$ and $[NK_N^{-1}]\ge N^{1-\delta_2}$. Then

$$\sum_{N=N'}^{\infty}P\{\Sigma(r,\lambda,N)<\lambda'\}\le\sum_{N=N'}^{\infty}\exp\{-N^{\delta_2}\}<\infty$$

by the integral test. Therefore, the Borel-Cantelli lemma asserts that, with probability one, only finitely many $\Sigma(r,\lambda,N)$ are less than λ'. Therefore

$$P\left\{\liminf_{N\to\infty}\Sigma(r,\lambda,N)\ge\lambda'\right\}=1.$$

Since $\lambda'=\lambda-\epsilon$, where $\epsilon>0$ is arbitrary, this means that

$$P\left\{\liminf_{N\to\infty}\Sigma_r(N,[(2\lambda^{-2}\log N)^{1/\alpha}])\ge\lambda\right\}=1,$$

and the theorem follows.

3. Comparison with the Erdös-Rényi theorem. The Erdös-Rényi theorem makes the following assertion:

Theorem 3.1 (Erdös-Rényi). *If* $\{X_n : 1 \le n < \infty\}$ *is a sequence of i.i.d. random variables with* $E(X_1) = 0$, $\mathrm{Var}(X_1) = 1$, *and m.g.f.* $\phi(t) < \infty$ *for a* $|t| < B$, *where* $0 < B \le \infty$, *then, for every* λ *in the range of* $Q(t) = \phi'(t)/\phi(t)$,

$$P\left\{\lim_{N\to\infty} \Sigma_1(N, [C(\lambda)\log N]) = \lambda\right\} = 1,$$

where $C(\lambda) = \{\lambda Q^{-1}(\lambda) - \log \phi(Q^{-1}(\lambda))\}^{-1}$.

We note that in going from Theorem 2.1 (which holds only for $1 < r < 2$) to Theorem 3.1 (which holds only for $r = 1$), the quantity $2\lambda^{-2}$ suddenly jumps to $C(\lambda)$. Why does that instantaneous jump occur? We observe that, by L'Hôpital's Rule,

$$\lim_{\lambda\to 0} 2\lambda^{-2}/C(\lambda) = 2 \lim_{\lambda\to 0} \lambda^{-2}\{\lambda Q^{-1}(\lambda) - \log\phi(Q^{-1}(\lambda))\}$$

$$= \lim_{\lambda\to 0} \lambda^{-1}Q^{-1}(\lambda) = (Q^{-1})'(0) = 1,$$

as $\mathrm{Var}(X_1) = 1$. This seems to say that as λn gets closer to $\lambda n^{1/r}$ (by the process of λ tending to 0), $C(\lambda)$ gets closer to $2\lambda^{-2}$, which would be its actual value if λn really were $\lambda n^{1/r}$ for $1 < r < 2$.

In the special case of normally distributed X_1, it turns out that $C(\lambda) = 2\lambda^{-2}$ for all values of λ, because $Q(t) = t$ in that case. We have therefore that

$$P\left\{\lim_{N\to\infty} \Sigma_r(N, [(2\lambda^{-2}\log N)^{1/\alpha}]) = \lambda\right\} = 1$$

for $1 \le r < 2$ in the normal case, where $\alpha = (2 - r)/r$. In the general case, however, for $r = 1$, denoting the value of $C(\lambda)$ in Theorem 3.1 by $2\theta^{-2}$ so that λ is $C^{-1}(2\theta^{-2})$, we can write

$$P\left\{\lim_{N\to\infty} \Sigma_1(N, [2\theta^{-2}\log N]) = C^{-1}(2\theta^{-2})\right\} = 1,$$

where $x = C^{-1}(2\theta^{-2})$ is the (unique) solution of the equation

$$xQ^{-1}(x) - \log\phi(Q^{-1}(x)) = \theta^2/2.$$

The solution exists if $C^{-1}(2\theta^{-2})$ lies in the range of Q. Now, because $C(\lambda)$ is a strictly decreasing function of λ, and $C(\lambda) < 2\lambda^{-2}$ for $\lambda > 0$, it follows that $C^{-1}(2\lambda^{-2}) > C^{-1}(C(\lambda)) = \lambda$. Under the conditions of Theorem 2.1, we then have with probability one that

$$\lim_{N\to\infty} \Sigma_r(N, [(2\lambda^{-2}\log N)^{1/\alpha}]) = \lambda \quad \text{for } 1 < r < 2,$$

and

$$\lim_{N\to\infty} \Sigma_1(N, [2\lambda^{-2}\log N]) = C^{-1}(2\lambda^{-2}) \quad \text{for } r = 1,$$

where, in the case of normality, $C^{-1}(2\lambda^{-2}) = \lambda$.

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REFERENCES

1. R. R. Bahadur and R. Ranga Rao, *On deviations of the sample mean*, Ann. Math. Statist. 31 (1960), 1015–1027. MR 22 #8549.

2. S. A. Book, *The Erdös-Rényi new law of large numbers for weighted sums*, Proc. Amer. Math. Soc. 38 (1973), 165–171. MR 46 #10044.

3. H. Cramér, *Sur un nouveau théorème-limite de la théorie des probabilités*, Actualités Sci. Indust., no. 736, Hermann, Paris, 1938, pp. 5–23.

4. P. Erdös and A. Rényi, *On a new law of large numbers*, J. Analyse Math. 23 (1970), 103–111. MR 42 #6907.

5. W. Feller, *An introduction to probability theory and its applications*. I, 3rd ed., Wiley, New York, 1968. MR 37 #3604.

6. I. A. Ibragimov and Yu. V. Linnik, *Independent and stationary sequences of random variables*, Wolters-Noordhoff, Groningen, 1971.

7. M. Loève, *Probability theory*, 3rd ed., Van Nostrand, Princeton, N. J., 1963. MR 34 #3596.

8. V. V. Petrov, *Generalization of Cramér's limit theorem*, Uspehi Mat. Nauk 9 (1954), no. 4 (62), 195–202. (Russian) MR 16, 378.

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