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ON THE PRODUCTS OF WEAKLY LINDELÖF SPACES

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ABSTRACT. The aim of this note is to show, without using any special set-theoretic assumptions, that the product of two (weakly) Lindelöf spaces is not necessarily weakly Lindelöf.

In [1] M. Ulmer has constructed two weakly Lindelöf spaces whose product is not so; in his construction, the assumption $2^{\aleph_0} = 2^{\aleph_1}$ was essentially employed. In this short note we shall provide another such example (where the factors are even Lindelöf), in the construction of which no additional set-theoretic assumption is used.

To start with, we shall deal with some properties of the "half-open" topologies on linearly ordered sets which may be interesting in themselves. We recall that, given a cardinal number α, X is (weakly) α-Lindelöf if every open cover of X has a subcover (weak subcover, i.e. a subfamily whose union is dense in X) of cardinality $\leq \alpha$.

Let $\langle R, \prec \rangle$ be a linearly ordered set. We shall denote by R^+ and R^-, respectively, the spaces on R for which the half-open intervals of the form $[x, y)$ and $(x, y]$, respectively, form an open basis.

Lemma 1. *Let α be an infinite cardinal number, and let $\langle R, \prec \rangle$ be an order complete linearly ordered set in which there is no decreasingly or increasingly ordered subset of type α^+. Then both R^+ and R^- are α-Lindelöf spaces.*

Proof. It will obviously suffice to show that R^+ is α-Lindelöf. To see this, let $\mathfrak{U}$ be a cover of R^+ by basic open sets of the form $[x, y)$. First we claim that for any $a, b \in R$, $a \prec b$, the segment $[a, b]$ can be covered by at most α members of $\mathfrak{U}$.

Indeed, using the completeness of $\langle R, \prec \rangle$, there is a $c \in [a, b]$ which is the least upper bound of those $d \in [a, b]$ for which the segment $[a, d]$ can be covered by $\leq \alpha$ members of $\mathfrak{U}$. We claim that $c = b$.

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Suppose, on the contrary, that $c < b$. The segment $[a, c]$ itself can be covered by $\leq \alpha$ members of $\mathfrak{U}$. This is obvious if $c = a$ or if c has an immediate predecessor. If not, it follows from the second condition on $\langle R, \prec \rangle$. Indeed, in this case we can take an increasing, well-ordered sequence $\langle c_\xi, \xi < \lambda \rangle$ converging to c from below, with $c_\xi \in (a, c)$ and $\lambda \leq \alpha$. By the choice of c, every segment $[a, c_\xi]$ can be covered by $\leq \alpha$ members of $\mathfrak{U}$; hence, so can $[a, c) = \bigcup \{[a, c_\xi]: \xi < \lambda\}$, and $[a, c]$ as well.

Now there are two cases to be distinguished.

First, if c has an immediate successor, say c', and $[x, y)$ is a member of $\mathfrak{U}$ containing c', then adding $[x, y)$ to any cover of $[a, c]$ with $\leq \alpha$ members of $\mathfrak{U}$ we obtain such a cover of $[a, c']$, contradicting the choice of c. Similarly if c has no immediate successor and $[x, y)$ is a member of $\mathfrak{U}$ containing c, then $[x, y)$ contains a $c' \succ c$, hence, adding it to an appropriate cover of $[a, c]$, we again get a contradiction.

Now making use again of the second condition on $\langle R, \prec \rangle$, we can obtain sequences $\langle a_\xi : \xi < \alpha \rangle$ and $\langle b_\xi : \xi < \alpha \rangle$ such that the first one is coinitial and the second is cofinal in $\langle R, \prec \rangle$. According to what we have proved above, every segment $[a_\xi, b_\eta]$ can be covered by $\leq \alpha$ members of $\mathfrak{U}$, hence so can

$$R^+ = \bigcup \{[a_\xi, b_\eta]: \xi, \eta < \alpha\}.$$

This completes the proof.

Lemma 2. *Let $\langle R, \prec \rangle$ be a densely ordered set with $d(R) > \beta$ (i.e., R does not contain a dense subset of cardinality $\leq \beta$). Then the product space $R^+ \times R^-$ is not weakly β-Lindelöf.*

Proof. Let us denote, as usual, by Δ the diagonal $\Delta = \{\langle p, p \rangle : p \in R\}$ of the product $R^+ \times R^-$, and put

$$\Gamma = \{\langle p, q \rangle \in R^+ \times R^- : p \prec q\}.$$

First we show that Γ is open in $R^+ \times R^-$. Indeed if $p \prec q$ then, by the denseness of $\langle R, \prec \rangle$, there is an r with $p \prec r \prec q$, and obviously $[p, r) \times (r, q]$ is a neighbourhood of $\langle p, q \rangle$ contained in Γ.

Thus

$$\mathfrak{U} = \{\Gamma\} \cup \{[p, \rightarrow) \times (\leftarrow, p]: p \in R\}$$

is an open cover of $R^+ \times R^-$, since for any $\langle p, q \rangle$ with $p \succeq q$ we have

$$\langle p, q \rangle \in [p, \rightarrow) \times (\leftarrow, p].$$

We claim that for any subfamily $\mathfrak{V} \subset \mathfrak{U}$ with $|\mathfrak{V}| = \beta$, the union $V = \bigcup \mathfrak{V}$ is not dense in $R^+ \times R^-$. Indeed, let us put

$$A = \{p \in R: [p, \rightarrow) \times (\leftarrow, p] \in \mathcal{O}\}.$$

Then $|A| \leq |\mathcal{O}| = \beta < d(R)$, so there is an open interval (a, b) of R such that $A \cap (a, b) = \emptyset$; and by the denseness of $\langle R, \prec \rangle$, there is a c with $a \prec c \prec b$.

Now the set $[c, b) \times (a, c]$ is obviously a nonempty open subset of $R^+ \times R^-$, and for any $p \in A$ we have

$$[p, \rightarrow) \times (\leftarrow, p] \cap [c, b) \times (a, c] = \emptyset,$$

since either $p \succeq b$ or $p \preceq a$. Moreover, we have, trivially,

$$\Gamma \cap [c, b) \times (a, c] = \emptyset.$$

This indeed shows

$$V \cap [c, b) \times (a, c] = \emptyset,$$

so that V is not dense; thus $R^+ \times R^-$ has an open cover with no weak subcover of cardinality $\leq \beta$, and consequently $R^+ \times R^-$ is not weakly β-Lindelöf.

Now we are ready to present our Example.

Example. Let $\langle R, \prec \rangle$ be any linearly ordered set satisfying the following conditions:

(i) $\langle R, \prec \rangle$ is continuously (i.e. both densely and completely) ordered;

(ii) $\langle R, \prec \rangle$ contains no uncountable decreasing or increasing well-ordered subset;

(iii) $d(R) = 2^{\aleph_0}$.

(The unit square with the lexicographic ordering is such an ordered set.) Then R^+ and R^- are Lindelöf spaces such that $R^+ \times R^-$ is not weakly Lindelöf, nor even weakly β-Lindelöf for any $\beta < 2^{\aleph_0}$.

The proof is obvious from Lemmas 1 and 2.

REFERENCE

1. M. Ulmer, *Products of weakly* $\aleph$*-compact spaces*, Trans. Amer. Math. Soc. 170 (1972), 279–284.

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