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MORSE-SMALE ENDOMORPHISMS OF THE CIRCLE

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ABSTRACT. The orbit structure of a continuously differentiable map f of the circle is examined, in the case where the nonwandering set of f is finite and hyperbolic. It is shown that there is a natural number $n(f)$ such that the period of any periodic point of f is $n(f)$ times a power of 2.

1. **Introduction.** It is well known (see [4]) that for a Kupka-Smale diffeomorphism f of the circle S^1 with $\Omega(f)$ finite, the following are true:

A. $\Omega(f)$ consists of periodic points.

B. The expanding and contracting periodic points alternate.

C. If f is orientation preserving, all periodic points have the same period, and if f is orientation reversing all periodic points have period one or two.

The purpose of this paper is to determine to what extent A, B, and C are true for a Kupka-Smale endomorphism f of S^1 with $\Omega(f)$ finite. (To avoid unnecessary confusion caused by certain pathological cases, we also assume a generic property about the singularities of f.) The results are stated in Theorems A, B, and C, following the necessary definitions.

We let $\mathrm{End}(S^1)$ denote the space of C^1 maps of S^1 into itself. Fix $f \in \mathrm{End}(S^1)$. A point $x \in S^1$ is said to be wandering if there is a neighborhood N of x in S^1 such that $f^i(N) \cap N = \varnothing$, $\forall i > 0$. The set of points which are not wandering is called the nonwandering set and denoted $\Omega(f)$. A point $x \in S^1$ is called a periodic point if $f^n(x) = x$ for some natural number n. The minimum of $\{n | f^n(x) = x\}$ is called the period of x.

A periodic point x of period n is said to be contracting if $|Df^n(x)| < 1$, and expanding if $|Df^n(x)| > 1$. f is said to be Kupka-Smale if any periodic point of f is expanding or contracting.

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A point $x \in S^1$ is called a singularity of f if $Df(x) = 0$. x is said to be eventually periodic if $f^m(x)$ is periodic for some natural number m, or equivalently if orb(x) is finite, where orb$(x) = \{f^n(x) | n \geq 0\}$.

We now define $MS(S^1)$ to be the set of $f \in \mathrm{End}(S^1)$ such that $\Omega(f)$ is finite, and:

(1) f is Kupka-Smale.

(2) No singularity of f is eventually periodic.

For $f \in MS(S^1)$ we let $\Omega_c(f)$ (respectively $\Omega_e(f)$) denote the set of contracting (respectively expanding) periodic points of f.

We will prove the following:

Theorem A. *If $f \in MS(S^1)$ then $\Omega(f)$ consists of periodic points.*

Theorem B. *Let $f \in MS(S^1)$, and* card *denote cardinality.*

$$\operatorname{card} \Omega_e(f) \leq \operatorname{card} \Omega_c(f) \leq \operatorname{card} \Omega_e(f) + 1.$$

Equality (*of* card $\Omega_e(f)$ *and* card $\Omega_c(f)$) *holds if and only if f is onto. In the onto case the expanding and contracting periodic points alternate.*

Theorem C. *Let $f \in MS(S^1)$. There is a natural number $n(f)$ such that the period of any periodic point of f is $n(f)$ times a power of* 2. (*Here we include* $1 = 2^0$ *as a power of* 2.)

We conclude this section with a few remarks. First, suppose $f \in MS(S^1)$ is C^2 and satisfies the additional generic properties:

(3) All singularities of f are nondegenerate (i.e. the second derivative is not zero).

(4) Orbits of distinct turning points are disjoint.

Then f is structually stable (see [1] or [3]). In fact, the set of maps f satisfying these properties can be classified up to topological conjugacy, by associating to each such f a finite diagram consisting of certain eventually periodic points of f and iterates of the singularities of f (see [1] for details, or [2] where a special case is studied).

Second, since $x \in \Omega(f) \Rightarrow f(x) \in \Omega(f)$, it is obvious that $f \in MS(S^1)$ and $x \in \Omega(f)$ imply orb(x) is finite. However this does not mean x is periodic for endomorphisms. So Theorem A is not immediate as it is in the diffeomorphism case.

Third, we note that one can construct (by induction) for any natural number n, a map f_n in $MS(S^1)$ with periodic points of period $1, 2, 4, \cdots, 2^n$ (see [1] for details). Thus the statement in Theorem C is essentially the most that can be said.

Finally, we remark that Theorems A, B, and C are true without the assumption that no singularity is eventually periodic. However, dropping this assumption makes a few of the proofs somewhat cumbersome, while adding little generality.

2. **Proof of Theorem A.** Let $f \in \mathrm{End}(S^1)$, and let x be an expanding periodic point of period n. We let $W_l^u(x)$ denote the local unstable manifold of x, which is simply an open interval about x on which $|Df^n| > 1$, such that $f^n(W_l^u(x)) \supset W_l^u(x)$. We set $W^u(x) = \mathrm{orb}(W_l^u(x))$, where $\mathrm{orb}(A)$ is defined for any set A by $\mathrm{orb}(A) = \bigcup_{n \geq 0} f^n(A)$.

We will use the following remark in the proof of Proposition 1. If g is a continuous map of S^1 into itself, and I is a closed interval in S^1 with $g(I) \supset I$ and $g(I) \neq S^1$, then g has a fixed point in I. This statement follows immediately from continuity (Rolle's theorem), but is false without the hypothesis $g(I) \neq S^1$.

Theorem A follows immediately from the following proposition.

Proposition 1. *Suppose $f \in \mathrm{End}(S^1)$ is Kupka-Smale and no singularity of f is eventually periodic. Suppose $y \in \Omega(F)$ is eventually periodic but not periodic. Then y is a limit of periodic points.*

Proof. By hypothesis there is an expanding periodic point p and an integer $k > 0$ with $f^k(y) = p$. Let V be any neighborhhod of y. By choosing V smaller if necessary, we may assume that $f^k(V)$ is a neighborhood of p in $W_l^u(p)$.

Note that $y \in \overline{W^u(p)}$ or else y would be wandering. But since $\overline{W^u(p)} - W^u(p)$ is a finite invariant set, we have $y \in W^u(p)$. Hence $\exists y_1 \in W_l^u(p)$ and $n > 0$ with $f^n(y_1) = y$. Let W be a closed interval about y_1 in $W_l^u(p)$ such that $f^n(W)$ is a neighborhood of y in V. Then $f^{n+k}(W)$ is a neighborhood of p in $W_l^u(p)$.

Now, there is a closed interval $K \subset f^{n+k}(W)$, and an integer $l > 0$, such that $f^l(K) = W$. So, $f^{n+k+l}(K) = f^{n+k}(W)$, which is a proper closed interval containing K. Hence K contains a periodic point, which implies that all iterates of K contain periodic points. In particular, since $V \supset f^n(W) = f^{n+l}(K)$, V has a periodic point. Since V was arbitrary this completes the proof. Q.E.D.

3. **Proof of Theorem B.**

Lemma 2. *Let $f \in MS(S^1)$ and let p be an expanding periodic point of f.*

There does not exist $y \in (\overline{W^u(p)} - \mathrm{orb}(p))$ *with* $p \in \mathrm{orb}(y)$.

Proof. Such an element y would be nonwandering, but not periodic, a contradiction by Theorem A. Q.E.D.

We now make another definition. We will use the notation $[a, b]$ to denote the arc from a to b in which b is in the counterclockwise direction from a. Let $f \in MS(S^1)$. Let p be an orientation preserving expanding fixed point of f. Set $W^u(p, cc) = \mathrm{orb}[p, b]$, where b is a point in $W^u_l(p)$ in the counterclockwise direction from p, and set $W^u(p, cl) = \mathrm{orb}[a, p]$, where a is a point in $W^u_l(p)$ in the clockwise direction from p. From the definition of $W^u_l(p)$, it follows that $W^u(p, cc)$ and $W^u(p, cl)$ are independent of the choices for a and b. If p is an orientation reversing expanding fixed point, define $W^u(p, cc)$ and $W^u(p, cl)$ by thinking of p as an orientation preserving fixed point of f^2. Finally, if p is an expanding periodic point of period n, define $W^u(p, cc)$ and $W^u(p, cl)$ by thinking of p as a fixed point of f^n.

Proposition 3. *Let p be an expanding periodic point of $f \in MS(S^1)$ and let $I = \overline{W^u(p, cc)}$ or $I = \overline{W^u(p, cl)}$. Then I is a proper subinterval of S^1 which contains another periodic point (besides p), and the closest periodic point to p in I is contracting.*

Proof. By looking at a power of f, we may assume without loss of generality that p is an orientation preserving fixed point. We may also assume that $I = \overline{W^u(p, cc)}$. If $I = S^1$, $\exists y \neq p$ in $W^u(p, cc)$ with $f(y) = p$. This contradicts Lemma 2. Hence I is a proper subinterval of S^1. Let $I = [p, b]$.

We put an order $<$ on I by identifying I with a subinterval of the real line. If $f(b) = b$ then b is a fixed point of f in I. If not $f(b) < b$. Since p is expanding, $\exists d \in W^u_l(p)$ in $[p, b]$ with $d < f(d)$. By continuity f has a fixed point in $[p, b]$.

Let c be the closest periodic point to p in I. We must show that c is contracting. Without loss of generality we may assume that c is an orientation preserving fixed point. Suppose c is expanding. $\exists l < c$, with $f(l) < l$. Hence there is a fixed point in $[d, l]$. This contradicts the fact that c is the closest periodic point to p in I. Hence c is contracting. Q.E.D.

If c is a contracting periodic point of period n of $f \in \mathrm{End}(S^1)$, we define the stable manifold of c by $W^s(c) = \{x \in S^1 | c \text{ is a limit point of } \mathrm{orb}(x)\}$. The component of $W^s(c)$ which contains c is called the semilocal stable manifold of c, and is denoted by $\mathrm{slsm}(c)$.

Proposition 4. *Let c be a contracting periodic point of $f \in MS(S^1)$. If*

$\mathrm{slsm}(c) \neq S^1$, *then one of the endpoints of* $\mathrm{slsm}(c)$ *is an expanding periodic point.*

Proof. Let E be the set of endpoints of $\mathrm{slsm}(c)$. E has one or two elements and $f^n(E) \subset E$, where c is of period n. Hence f has a periodic point in E. We show that any periodic point $p \in E$ is expanding. Suppose p is contracting. We may assume that c and p are orientation preserving fixed points, and p is in the counterclockwise direction from c. Put an order $<$ on $[c, p]$ as in Proposition 3. $\exists a$ and b in (c, p) with $f(a) < a$ and $b < f(b)$. Hence there is a fixed point in (c, p). This is a contradiction since c is the only fixed point in $\mathrm{slsm}(c)$. Q.E.D.

The following proposition follows almost immediately from the Lefschetz trace formula (see [6]).

Proposition 5. *If* $f \in MS(S^1)$ *then the degree of* f *is* $0, +1$, *or* -1. *If the degree of* f *is* 0, *then* $\mathrm{card}\,\Omega_c(f) = \mathrm{card}\,\Omega_e(f) + 1$. *If the degree of* f *is* ± 1 *then* $\mathrm{card}\,\Omega_c(f) = \mathrm{card}\,\Omega_e(f)$.

Proposition 6. *Let* $f \in MS(S^1)$ *be onto. Then* $\mathrm{card}\,\Omega_e(f) = \mathrm{card}\,\Omega_c(f)$.

Proof. Without loss of generality we may assume that all the periodic points of f are orientation preserving fixed points. Suppose the statement is false. Then there are two contracting fixed points c_1 and c_2 such that the interval (c_1, c_2) contains no fixed points. (The only other possibility is that $\Omega(f)$ consists of a single fixed sink c, but this would imply f is not onto by Proposition 4.)

Let $I = [c_1, c_2]$. Pick points $t_1 \in \mathrm{slsm}(c_1)$ and $t_2 \in \mathrm{slsm}(c_2)$ in I, such that $f(t_1) \in (c_1, t_1)$ and $f(t_2) \in (t_2, c_2)$. Let $J = [t_1, t_2]$. Then $f(J) \supset [f(t_2), f(t_1)]$. (For, if $f(J)$ did not contain this interval, it would have to contain $[f(t_1), f(t_2)]$. Then $f(J) \supset J$ and $f(J)$ is a proper subinterval of S^1. Hence there is a fixed point in J, a contradiction.)

Let $\Omega_e(f) = \{e_1, \cdots, e_n\}$. There are points $k_1, \cdots, k_n$ in J such that $f(k_i) = e_i$ for $i = 1, \cdots, n$. Since f is onto for each $i = 1, \cdots, n$, we can find a sequence (k_i^{-m}) with $k_i^0 = k_i$ and $f(k_i^{-m}) = k_i^{-m+1}$ $\forall m > 0$. The sequence (k_i^{-m}) must have a limit point, and a limit point of this sequence is clearly nonwandering. So to each k_i we can assign an expanding fixed point e_j such that e_j is a limit point of the sequence (k_i^{-m}). Define a map T: $\{k_1, k_2, \cdots, k_n\} \to \{k_1, k_2, \cdots, k_n\}$ by $T(k_i) = k_j$, where e_j is the chosen limit point of (k_i^{-m}). Any map from a finite set into itself has a periodic point, so there is a subset of $\{k_1, k_2, \cdots, k_n\}$, say $\{k_{j_1}, \cdots, k_{j_r}\}$, such that $T(k_{j_i}) = k_{j_{i+1}}$ for $i = 1, \cdots, r-1$ and $T(k_{j_r}) = k_{j_1}$.

Let U be any neighborhood of k_{j_1}. $T(k_{j_r}) = k_{j_1}$ means that e_{j_1} is a limit point of $(k_{j_r}^{-m})$. Now $f(U)$ is a neighborhood of e_{j_1} so some iterate of U contains k_{j_r}. Then $T(k_{j_{r-1}}) = k_{j_r}$ means e_{j_r} is a limit point of $(k_{j_{r-1}})$. So some iterate of U contains $k_{j_{r-1}}$. It follows after $r-2$ more steps that an iterate of U contains k_{j_1} and hence intersects U. Since U was arbitrary, k_{j_1} is nonwandering. This is a contradiction and completes the proof. Q.E.D.

Theorem B now follows from Propositions 5 and 6 (and the fact that if f is not onto then the degree of f is 0). In view of Proposition 5, the following corollary is essentially a restatement of the content of Theorem B.

Corollary 7. *If $f \in MS(S^1)$ and the degree of f is 0 then f is not onto.*

4. Proof of Theorem C.

Proposition 8. *Let e and c be adjacent expanding and contracting periodic points of $f \in MS(S^1)$ with c fixed. Then e is fixed by f^2.*

Proof. Without loss of generality we may assume that there are no periodic points in (e, c). Let e_1 be the closest point to e in the counterclockwise direction from e, in orb(e). We have two cases.

Case 1. $f(e) \neq e_1$. Then $f([e, c])$ contains c and the point $f(e)$ which is not in $[e, e_1]$. Hence $\exists x \in (e, c)$ such that $f(x) = e$ or $f(x) = e_1$. In either case $e \in$ orb(x), a contradiction by Lemma 2 and Proposition 3.

Case 2. $f(e) = e_1$. Note $[e, e_1] \subset W^u(e)$, because $f([e, c])$ is an interval containing c and e_1, so $f([e, c]) \supset [c, e_1]$. If $f(e_1) = e$ we are done, so suppose $f(e_1) \neq e$. Then $f([e, e_1])$ contains c, and the point $f(e_1)$ is not in $[e, e_1]$. Hence $\exists y \in (e, e_1)$ such that $f(y) = e$ or $f(y) = e_1$. In either case $e \in$ orb(y), a contradiction by Lemma 2.

Proposition 9. *Let e and c be adjacent expanding and contracting periodic points of $f \in MS(S^1)$ with e fixed. Then c has period a power of* 2. *(Here we include $1 = 2^0$ as a power of* 2.*)*

Proof. Suppose not. Without loss of generality we may assume that there are no periodic points in (e, c). Let p be the closest periodic point to e, in the counterclockwise direction from e, which has period a power of 2 (there is such a p by the proof of Proposition 3). Suppose p is of period

$k = 2^n$. If we let $g = f^{2k}$, then in the interval $[e, p]$, g has only two fixed points, e and p, both of which are orientation preserving. It follows that p is contracting. For if p is expanding, then by the proof of Proposition 3, $p \in W^u(e, cc)$ and $e \in W^u(p, cl)$. This implies that there is a nonperiodic nonwandering point in $W^u(e, cc)$, a contradiction.

Let b be the closest periodic point to p in (e, p). Then b is expanding by the proof of Theorem B, since $[b, p]$ is in $\mathrm{Im}(g)$. Under g, p is a contracting fixed point, and b is an expanding periodic point of period greater than 2. This contradicts Proposition 8. Q.E.D.

Theorem C now follows from Propositions 8 and 9 and Theorem B.

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