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HORIZONTAL CHORDS OF THE GRAPH OF A CONTINUOUS FUNCTION

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ABSTRACT. The continuous function f, defined on $[a, b]$, changing sign exactly n times on (a, b), and $f(a) = f(b) = 0$, has horizontal chords of every length $h < H$ with endpoints in (a, b). We determine the largest such H as a function of n.

By a theorem of R. J. Levit [1], we know that if f is a continuous real function on the closed interval $[a, b]$, changes sign exactly n times in the open interval (a, b), and $f(a) = f(b) = 0$, then the graph of f has horizontal chords of every length $h \leq H_1 = (b - a)/[(n+3)/2)]$ with endpoints in $[a, b]$, i.e., for every such h at least one $x \in [a, b - h]$ exists such that $f(x + h) = f(x)$. In case of endpoints in $(a, b]$, H_1 reduces to $H_2 = (b - a)/[(n + 5)/2]$. These two bounds cannot be improved.

It is easy to see that H_2 remains the best possible bound if the endpoints are in $[a, b)$. So far the bounds hold for n both even and odd. The purpose of this note is to consider the situation when we ask for horizontal chords with endpoints in (a, b).

Theorem. *If f is a continuous real function on $[a, b]$, changes sign exactly n times in (a, b), and $f(a) = f(b) = 0$, then for any positive h such that*

$$h \leq H_2 = \frac{b-a}{[(n+5)/2]} = \frac{b-a}{(n+4)/2} \qquad \text{for } n \text{ even } (n > 0),$$

and

$$h \leq H_3 = \frac{b-a}{[(n+7)/2]} = \frac{b-a}{(n+7)/2} \qquad \text{for } n \text{ odd},$$

there is a number $x \in (a, b - h)$ such that $f(x + h) = f(x)$. H_2 and H_3 cannot be improved.

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Proof. 1. The following three cases, (a), (b), and (c), hold for n even as well as for n odd.

(a) If $f(a+h) \neq 0$ and $f(b-h) \neq 0$, the statement follows at once from Levit's theorem, as it does in the cases:

(b) $f(a+h)=0$, $f(b-h) \neq 0$; and

(c) $f(a+h) \neq 0$, $f(b-h)=0$.

We remark that up to this point for every $h \leq (b-a)/[(n+5)/2]$, there is a number $x \in (a, b-h)$ such that $f(x) = f(x+h)$.

2. Let $f(a+h) = f(b-h) = 0$, and let n^* be the number of times f changes sign in $(a+h, b-h)$; then for both even and odd n

$$h \leq (b-a)/[(n+7)/2] \Leftrightarrow h\{[(n+3)/2]+2\} \leq b-a.$$

Hence,

$$h \leq \frac{b-a-2h}{[(n+3)/2]} \leq \frac{(b-h)-(a+h)}{[(n^*+3)/2]},$$

and according to Levit's theorem there is a number x such that

$$a < a+h \leq x \leq b-2h < b-h,$$

i.e., $x \in (a, b-h)$ anyway, and $f(x+h) = f(x)$. For n odd the first part of the Theorem is proved, while for n even we still have to improve the bound $(b-a)/[(n+7)/2]$, gained under the assumption $f(a+h) = f(b-h) = 0$.

3. Let n be even and $n^* \leq n-2$. Thus,

$$h \leq \frac{b-a}{[(n+5)/2]} \Leftrightarrow h \leq \frac{b-a-2h}{[(n+1)/2]},$$

and since $n^* \leq n-2$, it follows that

$$h \leq \frac{(b-h)-(a+h)}{[(n^*+3)/2]}.$$

The existence of the required number x is again a consequence of Levit's theorem.

4. Before we continue, let us give a definition. We say that f has a *true zero* in x if f has a zero in x, but does not change sign in x.

Let n be even. Assuming $n^* = n$ or $n^* = n-1$, we still have to prove our statement for the following five cases, respective to the properties of f;

(a) $n^* = n$, and true zeros in $a+h$ and $b-h$;

(b) $n^* = n-1$, true zeros in $a+h$ and $b-h$, and one change of sign in $(a, a+h)$;

(c) $n^* = n-1$, true zeros in $a+h$ and $b-h$, and one change of sign in $(b-h, b)$;

(d) $n^* = n-1$, true zero in $b-h$, and a change of sign in $a+h$;

(e) $n^* = n - 1$, true zero in $a + h$, and a change of sign in $b - h$.

We remark that in (a), (d), and (e), $f(x) \geq 0$ [$f(x) \leq 0$, respectively] for every $x \in (a, a + h) \cup (b - h, b)$, and in (b) and (c), $f(x) \geq 0$ [$f(x) \leq 0$, respectively] in one of the intervals $(a, a + h)$ or $(b - h, b)$, while in the other one f changes sign.

In the following two lemmas we assume without loss of generality the case not in brackets in the preceding paragraph.

Lemma 1. *If f satisfies* (a), (d), *or* (e), *then for every positive $h < (b - a)/2$ there exists a number $x \in (a, b - h)$ such that $f(x + h) = f(x)$.*

Proof of Lemma 1. Suppose for some $h < (b - a)/2$ such x does not exist. Then for $x \in (a, b - h)$, either (i) $f(x) < f(x + h)$ or (ii) $f(x) > f(x + h)$ is true. If (i) holds there is an integer $m > 0$ satisfying $((b - h) - mh) \in (a, a + h)$. It follows that

$$0 = f(b - h) > f((b - h) - mh) \geq 0,$$

which is a contradiction. If (ii) holds, there is an integer $m > 0$ satisfying $((a + h) + mh) \in (b - h, b)$. Thus, the contradiction

$$0 = f(a + h) > f((a + h) + mh) \geq 0$$

completes the proof of Lemma 1.

Lemma 2. *If f satisfies* (b) *or* (c) *– without loss of generality we choose* (b) *– then for every positive $h < (b - a)/2$ there is a number $x \in (a, b - h)$ such that $f(x + h) = f(x)$.*

Before starting the proof we give a short summary of the properties of the used function f. For $x \in (b - h, b)$ $f(x) \geq 0$ holds; f changes sign at $x_0 \in (a, a + h)$; $f(x) \geq 0$ for $x \in (a, x_0]$ since n is even, consequently, $f(x) \leq 0$ for $x \in [x_0, a + h]$, and since f has a true zero in $a + h$, $f(x) \leq 0$ for $x \in [x_0, a + h + \epsilon)$, where $\epsilon > 0$.

Proof of Lemma 2. Suppose $h < (b - a)/2$ and the required x does not exist. It follows that either $f(x) < f(x + h)$ or $f(x) > f(x + h)$ for $x \in (a, b - h)$. Let $f(x) < f(x + h)$; $f(x) \geq 0$ for $x \in (a, x_0]$ implies $f(x + h) > 0$ for $x + h \in (a + h, x_0 + h]$. This is a contradiction to the fact $f(x) \leq 0$ for $x \in (a + h - \epsilon, a + h + \epsilon)$, where $\epsilon > 0$, shown in the summary above. If $f(x) > f(x + h)$, the proof is the same as in Lemma 1. □

Let $n > 0$, then

$$\frac{b - a}{(n + 4)/2} < \frac{b - a}{2}.$$

According to the two lemmas, this completes the proof of the first part of the Theorem for n even.

5. To see that H_j, $j = 2, 3$, are the best possible bounds, we present two polygonal arcs R_j (R_2 was already used by Levit) satisfying the conditions of our Theorem and with the property that for any integer $n > 0$ and any $H > H_j$, there is a number $h < H$ such that $R_j(x + h) - R_j(x) \neq 0$ for every $x \in (a, b - h)$.

We choose

$$(*) \qquad h \in (H_j, \min\{H_{j-1}, H\}),$$

set $m = [(b - a)/h]$, $a_i = a + ih$, $b_i = b - ih$ $(i = 0, 1, \cdots, m)$, and $b_i' = b_0' - ih$ $(i = 1, 2, \cdots, m)$, where

$$b_0' = b - wh/(m + 1) \quad \text{and} \quad w = (b - a)/h - m.$$

Thus, the numbers a_i, b_i, and b_i' form finite sequences

$$a = a_0 < b_m < a_1 < \cdots < b_1 < a_m < b_0 = b$$

and

$$a = a_0 < b_m' < a_1 < b_{m-1}' < \cdots < a_m < b_0' < b.$$

We proceed with our construction by defining points

$$P_0(a, 0), \quad P_i(a_i, i - 1), \quad i = 1, 2, \cdots, m,$$

$$Q_m(b, 0), \quad Q_i(b_{m-i}, 1 + i - m), \quad i = 0, 1, \cdots, m - 1,$$

$$Q_i'(b_{m-i}', i - m - w/(1 - w)), \quad i = 0, 1, \cdots, m,$$

and polygonal arcs

$$R_2(x){:}\ P_0Q_0'P_1Q_1' \cdots Q_{m-1}'P_mQ_m'Q_m \quad \text{for } n \text{ even, and}$$

$$R_3(x){:}\ P_0Q_0P_1Q_1 \cdots Q_{m-1}P_mQ_m \quad \text{for } n \text{ odd.}$$

Since each of the points Q_i' $(i = 0, 1, \cdots, m)$, and Q_i $(i = 0, 1, \cdots, m - 2)$ lies below, and each of the points P_i $(i = 2, 3, \cdots, m)$, lies above the x-axis, while P_1 and Q_{m-1} lie on the x-axis in the interval (a, b), R_2 changes sign exactly $(m - 1) + (m - 1) = 2m - 2$ times, while R_3 changes sign exactly $(m - 2) + (m - 3) = 2m - 5$ times. By $(*)$,

$$H_j < h < H_{j-1} \Leftrightarrow [(n - 1 + 2j)/2] < (b - a)/h < [(n + 1 + 2j)/2],$$

which implies $m = [(n - 1 + 2j)/2]$, since

$$[(n - 1 + 2j)/2] + 1 = [(n + 1 + 2j)/2].$$

If $j = 2$, then $m = [(n+3)/2]$, and since n even, $n = 2m - 2$; if $j = 3$, then $m = [(n+5)/2]$, and since n odd, $n = 2m - 5$. Thus, R_2 and R_3 have the required number of n changes of sign. Since polygonal arcs are continuous, $P_0 = P_0(a, 0)$, and $Q_m = Q_m(b, 0)$, R_2 and R_3 satisfy the conditions of the Theorem.

It remains to prove that R_2 and R_3 have no chords of length h with endpoints in (a, b). But because of the construction of these polygonal arcs, simple reasoning shows that $R_2(x+h) - R_2(x) = 1$ for $x \in [b_m', b-h)$. If $x \in [a, b_m']$, the difference $R_2(x+h) - R_2(x)$ is strictly increasing from 0 to 1, assuming these values just at the endpoints of $[a, b_m']$. Thus, $R_2(x+h) - R_2(x) \neq 0$ for every $x \in (a, b-h)$. Quite alike, $R_3(x+h) - R_3(x) = 1$ for $x \in [b_m, a_{m-1}]$. The difference $R_3(x+h) - R_3(x)$ is strictly increasing from 0 to 1 for $x \in [a, b_m]$, and strictly decreasing from 1 to 0 for $x \in [a_{m-1}, b-h]$, assuming 0 only in a and $b-h$. Hence, $R_3(x+h) - R_3(x) \neq 0$ for $x \in (a, b-h)$. This completes the proof of the Theorem.

What happens when $n = 0$ (the case omitted in the above Theorem)? Items 1 and 2 of the proof hold for $n = 0$ as well as for $n > 0$, while in the third item, $n = 0$ is impossible. In item 4 for $n = 0$, the only case possible is $n^* = n$. Since Lemma 1 holds for $n = 0$, we come to the following result.

Corollary. *If f satisfies the conditions of the Theorem, and $n = 0$, then for every positive h such that $h < (b-a)/2$, there is a number $x \in (a, b-h)$ such that $f(x+h) = f(x)$.*

We cannot improve this result, since there is a function R satisfying the conditions of the Theorem and $R(x+h) - R(x) \neq 0$ for $h = (b-a)/2$ and $x \in (a, b-h)$. If $h = (b-a)/2$, then $a + h = b - h = (a+b)/2$. Let $R((a+b)/2) = 0$. The function

$$R(x) = \begin{cases} -(x-a)(x-(a+b)/2) & \text{for } a \leq x \leq (a+b)/2, \\ -2(x-b)(x-(a+b)/2) & \text{for } (a+b)/2 \leq x \leq b \end{cases}$$

is continuous, $R(a) = R((a+b)/2) = R(b) = 0$, and does not change sign in (a, b). If $x \in (a, (a+b)/2)$, then $x + h \in ((a+b)/2, b)$. Since $h = (b-a)/2$,

$$R(x+h) - R(x) = -(x-(a+b)/2)(x-a) \neq 0$$

for $x \in (a, (a+b)/2)$, and R has no chords of length $(b-a)/2$ with endpoints in (a, b).

If throughout the foregoing, the condition of n changes of sign is replaced by n zeros in (a, b), we come to the following results, whereof (1) has already been remarked by Levit.

(1) *For any* $h \leq (b - a)/[(n + 3)/2]$ *there is a number* $x \in [a, b - h]$;

(2) *for any* $h \leq (b - a)/[(n + 4)/2]$ *there is a number* $x \in [a, b - h)$ ($x \in (a, b - h]$, *respectively*);

(3) *for any* $h \leq (b - a)/[(n + 5)/2]$ *there is a number* $x \in (a, b - h)$ *such that* $f(x) = f(x + h)$.

The bounds given above are again the best possible ones; R_2 can be directly used in (2) for n odd and $x \in (a, b - h]$, and R_3 in (3) for n odd. The other polygonal arcs can be constructed likewise.

REFERENCE

1. R. J. Levit, *The finite difference extension of Rolle's theorem*, Amer. Math. Monthly 70 (1963), 26–30. MR 26 #5107

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