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SHORTER NOTES

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THE ZARISKI-LIPMAN CONJECTURE FOR HOMOGENEOUS COMPLETE INTERSECTIONS

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ABSTRACT. A new short proof is given that if R is a homogeneous complete intersection over a field K of char 0 and $\mathrm{Der}_K(R, R)$ is R-free, then R is a polynomial ring.

Let K be a field with char $K = 0$. The Zariski-Lipman conjecture asserts that if R is the local ring at a closed point y of a K-variety and $\mathrm{Der}_K(R, R)$ is R-free, then y is a simple point, that is, R is regular. This is clearly an affine question. The homogeneous case arises when I is a homogeneous prime in $S = K[x_1, \ldots, x_n]$ and R is the local ring of S/I at $y = (0, \ldots, 0)$. Then localization does not affect the issues and if we simply let $R = S/I$ instead, the conjecture is that if $\mathrm{Der}_K(R, R)$ is R-free, then R is a polynomial ring over K, i.e. I is generated by 1-forms. In [3] it is shown in the general case that if $\mathrm{Der}_K(R, R)$ is free then R is integrally closed. In [4] S. Moen showed that if R is a homogeneous complete intersection, i.e. I is generated by an S-sequence of forms $f_1, \ldots, f_r$, the conjecture holds. A different, shorter proof of Moen's result follows.

Assume that $I = (f_1, \ldots, f_r)S$ as above is prime in $S = K[x_1, \ldots, x_n]$, that the f_i are forms, that $d = \dim R = n - r$, and that $\mathrm{Der}_K(R, R)$ is free. We must show that $f_1, \ldots, f_r$ are 1-forms. We may assume, as usual, that $d_j = \deg f_j \geq 2$ for each j, and we shall obtain a contradiction. We identify $\mathrm{Der}_K(R, R)$ with the R-relations on the rows of the matrix $J = (\overline{f_{ji}})$, where $f_{ji} = \partial f_j/\partial x_i$ and $^-$ denotes reduction mod I. Thus, J is an n by r matrix. Since K is infinite, by a suitable K-linear change of coordinates we may arrange that

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$$(*) \qquad f_1, \ldots, f_r, x_{r+1}, \ldots, x_n$$

is a homogeneous system of parameters for S.

Let U be a matrix whose rows are a free basis for the R-relations on the rows of J. Since rank J = height $I = r$, U will be d by n. Moreover, we may assume that the first row of U is $(x_1^- \ldots x_n^-)$ (this is a relation by (#) below, and is part of a minimal basis by a degree argument). Then $0 \longrightarrow R^d \xrightarrow{U} R^n \xrightarrow{J} R^r$ is exact. If A is a matrix let $D(A)$ denote the determinant of the r by r submatrix in the upper left-hand corner. It follows from [1, Theorem 3.1] that $D(J)$ is a multiple of the rightmost d by d minor of U, and hence lies in the ideal $(x_{r+1}^-, \ldots, x_n^-)R$, so that

$$D(f_{ji}) \in (x_{r+1}, \ldots, x_n, f_1, \ldots, f_r)S.$$

Let * denote reduction mod $(x_{r+1}, \ldots, x_n)S$. Then

$$(**) \qquad D = D(f_{ji}^*) \in (f_1^*, \ldots, f_r^*)S^*,$$

where $S^* \cong K[x_1, \ldots, x_r]$. Since

$$(\#) \qquad d_j f_j = \sum_{i=1}^{n} f_{ji} x_i, \qquad 1 \leq j \leq r,$$

we have $f_j^* = (1/d_j)g_j$ where $g_j = \sum_{i=1}^r f_{ji}^* x_i$, $1 \leq j \leq r$. The g_j (or f_j^*) are a system of parameters for S^* (by (*) above) and by [2, first paragraph, proof of Theorem 1, pp. 227–228] the image of D generates the socle of the 0-dimensional Gorenstein local ring $S^*/(g_1, \ldots, g_r)S^*$, which contradicts (**) above. Q.E.D.

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