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A CHARACTERIZATION OF STEINITZ GROUP RINGS

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ABSTRACT. A ring R with an identity is a (right) *Steinitz* ring provided any linearly independent subset of a free (right) R-module can be extended to a basis for the module by adjoining elements from any given basis. In this paper, we characterize those group rings which are Steinitz rings by the following:

Theorem. *The group ring* $R[G]$ *is a Steinitz ring if and only if* R *is a Steinitz ring and either* (1) char $R = p^i$ *and* G *is a finite* p*-group or* (2) char $R = 0$ *and* $G = 1$.

Introduction. A ring R with an identity will be called a (right) *Steinitz* ring provided any linearly independent subset of a free (right) R-module can be extended to a basis for the module by adjoining elements from any given basis. A subset S of the ring R will be called *T-nilpotent* if for each sequence $\{x_i\}_{i=1}^{\infty}$ in S, there exists an integer n such that $x_n x_{n-1} \cdots x_1 = 0$. Chwe and Neggers [1], [2] proved that R is a Steinitz ring if and only if R is a local ring (i.e., the Jacobson radical is the set of nonunits) with a T-nilpotent Jacobson radical. In this paper we characterize those group rings which are Steinitz rings.

A characterization of Steinitz group rings. Since Steinitz rings have characteristic 0 or p^i where p is a prime, our characterization consists of two cases and is stated as follows:

Theorem. *The group ring* $R[G]$ *is a Steinitz ring if and only if* R *is a Steinitz ring and either* (1) char $R = p^i$ *and* G *is a finite* p*-group or* (2) char $R = 0$ *and* $G = 1$.

Proof. Suppose that $R[G]$ is a Steinitz group ring. Since the map ν defined by $(\Sigma r_g g)\nu = \Sigma r_g$ is a homomorphism from $R[G]$ onto R, it follows that R is a Steinitz ring. Since $(1 - g)\nu = 0$, it is clear that $1 - g$ is a nonunit for every $g \in G$. Consequently, $\{1 - g \mid g \in G\} \subseteq J(R[G])$ where $J(R[G])$ denotes the radical of $R[G]$. When $\alpha = \Sigma r_g g$ is an element of $R[G]$, the support of α will mean $\{g \in G \mid r_g \neq 0$ in the representation $\alpha = \Sigma r_g g\}$ and r_1 will

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be called the trace of α. When $g_1, g_2, \cdots, g_{j-1}$ are elements of G, let S_j denote the support of $(1 - g_{j-1})(1 - g_{j-2}) \cdots (1 - g_1)$. Suppose $g_1, g_2, \cdots, g_{n-1}$ have been chosen from G such that the trace of $(1 - g_{n-1})(1 - g_{n-2}) \cdots (1 - g_1)$ is 1. If $G_n = G - (S_n \cup S_n^{-1}) \neq \emptyset$, then for any $g_n \in G_n$, it follows that $(1 - g_n)(1 - g_{n-1}) \cdots (1 - g_1)$ has trace 1, hence the product is nonzero. Since $J(R[G])$ is T-nilpotent, $G_n = \emptyset$ for some n and it follows that G is a finite group.

The radical of a Steinitz ring contains all of the nonunits, thus for each element g, either g or $1 - g$ is a unit. It follows that the only idempotents in a Steinitz ring are 0 and 1. On the other hand, if g is an element of order n in the group G and if n is a unit in $R[G]$, then $n^{-1}(1 + g + \cdots + g^{n-1})$ is idempotent, hence $n^{-1}(1 + g + \cdots + g^{n-1})$ must be 0 or 1. When char $R = 0$, n is always a unit and thus the finite group G must be trivial. When char $R = p^i$, it must follow for each $g \in G$ that $n = 1$ or $p|n$. In this case, G is a p-group.

If R is a Steinitz ring and condition 2 holds, it is obvious that $R[G]$ is a Steinitz ring. When R is a Steinitz ring and condition 1 holds, we will show $R[G]$ is a Steinitz ring with the aid of the following:

Lemma. *Let R be a ring of characteristic p^i and let $1 = Z_0 \subset Z_1 \subset \cdots \subset Z_m = G$ be the ascending central series of the finite p-group G. For each $r = 0, 1, \cdots, m$ there exists an integer n_r such that any sequence in $\{1 - g | g \in G\}$ with at least n_r terms of the form $1 - z$ with $z \in Z_r$ has product zero in $R[G]$.*

Proof. For any product s of factors $1 - x_i$, $x_i \in G$, we define $\gamma_t(s)$ to be the number of factors where $x_i \in Z_t$. When $\gamma_0(s) \geq 1$, it is clear that $s = 0$. We may take $n_0 = 1$. Suppose n_r is a number such that $\gamma_r(s) \geq n_r$ implies $s = 0$. Let $n_{r+1} = n_r + n_r |Z_{r+1}| ip^i |G|$. We will show $\gamma_{r+1}(s) \geq n_{r+1}$ implies $s = 0$. The Lemma will follow by induction because of the nilpotence of G.

We shall use a second induction step. Namely, it will be shown that if $0 \leq k \leq n_r$ and s is a product with $\gamma_r(s) \geq k$ and $\gamma_{r+1}(s) \geq n_{r+1} - k$, then s is a sum of products s_i for which $\gamma_r(s_i) \geq k + 1$ and $\gamma_{r+1}(s_i) \geq n_{r+1} - (k + 1)$. Repeating this step at most n_r times will show that if $\gamma_{r+1}(s) \geq n_{r+1}$, then s is a sum of products s_i for which $\gamma_r(s_i) \geq n_r$, and consequently $s = 0$.

Suppose that $\gamma_r(s) \geq k$ and $\gamma_{r+1}(s) \geq n_{r+1} - k$, where $0 \leq k \leq n_r$. It follows from a pigeonhole argument that s contains at least $n_r ip^i |G|$ equal factors $1 - x$ for some $x \in Z_{r+1}$, since

$$\frac{n_{r+1} - k}{|Z_{r+1}|} \geq \frac{n_{r+1} - n_r}{|Z_{r+1}|} = n_r i p^i |G|.$$

There is nothing to prove if $s = 0$, so we may suppose that $\gamma_r(s) < n_r$. Therefore, there must be a product π of consecutive factors in s which contains $n = ip^i|G|$ of the equal factors $1 - x$ and does not contain any factor $1 - y$ for $y \in Z_r$. Restricting our attention to this subproduct π, we reorder the factors of π to collect the equal factors $1 - x$, using the relation

$$(1 - u)(1 - x) = (1 - x)(1 - u) + ux(1 - x^{-1}u^{-1}xu).$$

Since $(1 - x)^n = 0$, we know s is zero plus a sum of products s_i where each s_i is obtained by replacing a product $(1 - u)(1 - x)$ in π by $ux(1 - x^{-1}u^{-1}xu)$. Because of our choice of π, we know $u \notin Z_r$ and $x \notin Z_r$. Since $x \in Z_{r+1}$, it follows that $x^{-1}u^{-1}xu \in Z_r$ and therefore $\gamma_r(s_i) = \gamma_r(s) + 1 \geq k + 1$. Since we have removed two factors from s and introduced the new element $1 - x^{-1}u^{-1}xu$, we have $\gamma_{r+1}(s_i) \geq \gamma_{r+1}(s) - 1 \geq n_{r+1} - (k + 1)$. The product s_i contains the factor ux, but this is of no consequence since it can be moved to the far right of s_i by using the relation $g(1 - y) = (1 - gyg^{-1})g$. This does not change the numbers γ_t since Z_t is normal in G, and the proof of the Lemma is complete.

We are now ready to proceed with the proof of the Theorem. Suppose R is a Steinitz ring of characteristic p^i and G is a finite p-group. When $S = \{1 - g | g \in G\}$, the Lemma implies there exists a positive integer $k = n_m$ such that $S^k = 0$. Let

$$N = \sum_{g \in G} R(1 - g).$$

Since $x(1 - g) = (1 - xg) - (1 - x)$, it is an easy matter to show N is an ideal in the ring $R[G]$. Let $\{x_1, \cdots, x_k\}$ be any set of k elements of N where $x_i = \Sigma_{g \in G} r_{ig}(1 - g)$. Since elements of R commute with elements of S, $x_k x_{k-1} \cdots x_1$ is clearly a sum of terms of the form

$$r(1 - g_1)(1 - g_2) \cdots (1 - g_k)$$

and hence $N^k = 0$.

Let $J(R)$ denote the radical of the Steinitz ring R. We know $J(R)$ is a T-nilpotent subset of R consisting precisely of the nonunits of R. In general, the sum of a T-nilpotent subring and a T-nilpotent ideal is T-nilpotent, and in our setting we argue as follows: Since $J(R) + N/N$ is a homomorphic image of $J(R)$, we know $J(R) + N/N$ is a T-nilpotent set. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $J(R) + N$. From the sequence $\{x_i + N\}$ in $J(R) + N/N$, we can choose integers $m_0 = 0, m_1, \cdots, m_k$ where $y_j = \Pi_{i=m_{j-1}+1}^{m_j} x_i \in N$ for $j = 1, 2, \cdots, k$. Clearly,

$$x_{n_k} x_{n_{k-1}} \cdots x_1 = y_k y_{k-1} \cdots y_1 = 0,$$

since $N^k = 0$ and it follows that $J(R) + N$ is a T-nilpotent subset of $R[G]$. Using the fact that $R[G] = R + N$, it is an easy matter to show $J(R) + N$ is an ideal in $R[G]$. Let $x \in R[G]$ where $x \notin J(R) + N$. Writing $x = u(1 - z)$, where u is a unit in R and $z \in N$, one concludes immediately that

$$x^{-1} = (1 + z + \cdots + z^{k-1})u^{-1},$$

since $N^k = 0$. Therefore, if $x \notin J(R) + N$, then x is a unit in $R[G]$ and it follows that $J(R) + N$ is the ideal of nonunits in $R[G]$. It has now been shown that $R[G]$ is a Steinitz ring of characteristic p^i when R is a Steinitz ring of characteristic p^i and G is a finite p-group.

Note. Since Steinitz rings are perfect rings, one can use a result of S. M. Woods [4] to prove that G is a finite group when $R[G]$ is a Steinitz ring. The direct proof given above depends on $R[G]$ being Steinitz rather than merely perfect. I. G. Connell (see [3, Theorem 9]) proved the fundamental (augmentation) ideal N is nilpotent when G is a finite p-group and p is nilpotent in R. Connell's proof of this result is by induction on the order of G and the proof only guarantees the existence of k such that $N^k = 0$. On the other hand, the proof of our Lemma gives a method by which a specific k can be calculated when needed. Consequently, we presented our Lemma as an alternative rather than quoting Connell's result. In addition, we pose the following:

Problem. When $I = Z_0 \subset Z_1 \subset \cdots \subset Z_m = G$ is the ascending central series of the finite p-group G and R is a ring of characteristic p^i, find a better bound, or the best bound, for the smallest integer k such that $N^k = 0$.

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