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ALMOST CONTINUITY OF THE CESÀRO-VIETORIS FUNCTION

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ABSTRACT. Consider the following function due to Cesàro: $\phi(0) = 0$, and if $0 < x \leq 1$,

$$\phi(x) = \lim\sup(a_1 + a_2 + \cdots + a_n)/n,$$

where the a_i are given by the unique nonterminating binary expansion of $x = (0.\ a_1a_2\cdots)$. Vietoris proved in 1921 that ϕ is connected (as a subset of $[0, 1]\times R$). The purpose of this note is to alter Vietoris's argument in order to prove that ϕ is actually almost continuous in the sense of Stallings, thus answering a question raised recently by B. D. Smith.

B. D. Smith [7] recently proved that a real function f: $[0, 1] \to R$ is continuous if and only if it satisfies the three conditions (i) f is *almost continuous in the sense of Stallings* [8] (for any open set $N \subset [0, 1] \times R$ containing f, N contains a continuous function g: $[0, 1] \to R$), (ii) f is *almost continuous in the sense of Husain* [3] (for every $x \in [0, 1]$ and open set $V \subset R$ containing $f(x)$, $\mathrm{Cl}(f^{-1}(V))$ is a neighborhood of x), and (iii) f is *not of Cesàro type* (for every open $U \subset [0, 1]$ and open $V \subset R$ there exists $y \in V$ such that $U \not\subset \mathrm{Cl}(f^{-1}(y))$). In investigating the possible redundancy of (i), (ii), and (iii), Smith leaves open only the question of whether (i) and (ii) imply (iii). He poses as a very likely counterexample to this implication the function ϕ of Cesàro. It satisfies (ii) but not (iii). Vietoris [9] showed that ϕ is connected, and Smith points out that Stallings [8] raised the question of whether connected functions from $[0, 1]$ into R necessarily satisfy (i). If such were the case, then ϕ would necessarily satisfy (i). However, this question of Stallings has been answered in the negative. The first counterexample, due to Jones and Thomas [4], happens to satisfy (iii), but later examples [1], [2], [6], satisfy (ii) but not (iii), are connected, but do not satisfy (i). This casts some doubt as to whether ϕ might satisfy (i).

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It might be said here that it is possible to use standard transfinite techniques to construct a function f which intersects every closed subset C of $[0, 1] \times R$ which has a nondegenerate connected projection onto $[0, 1]$. Then, f would satisfy (ii) but not (iii), and it would follow from the theorem about "minimal blocking sets" of [5] that f would satisfy (i). However, it seems desirable to just determine whether the simpler and oft-used function ϕ satisfies (i).

Theorem. *The Cesàro function ϕ is almost continuous in the sense of Stallings.*

Proof. Vietoris's proof that ϕ has a connected graph [9, pp. 202–204] will be referred to extensively. Suppose ϕ is not almost continuous in the sense of Stallings, and let O be an open subset of $[0, 1] \times R$ containing ϕ but containing no continuous function $g: [0, 1] \to R$. Let N be a neighborhood of $(0, 0)$ which lies in O, assume without loss of generality that $N \cap Y = O \cap Y$ (Y is the y-axis), and define $G = \{(x, y) | (x, y) \in N$ or else $0 < x$ and there exists a continuous function $g: [0, x] \to R$ lying in O such that $g(x) = y\}$. G is obviously connected. It is also open, for suppose $(x, y) \in G$ and $g: [0, x] \to R$ is a continuous function lying in O such that $g(x) = y$. Consider a rectangle s with center (x, y), interior lying in O, and having a vertical left edge that intersects g in a point (x', y'). Clearly the function $g' = g | [0, x']$ can be extended continuously within O to any point interior to s, so that the interior of s lies in G. Let H denote the boundary of G and $J = [0, 1] \times R - (G \cup H)$. Let $b = \sup \{x | (x, \phi(x)) \in G\}$. Clearly $b > 0$. Suppose $(b, \phi(b))$ belongs to G and let $g: [0, b] \to R$ be a continuous function lying in O such that $g(b) = \phi(b)$. Then $b < 1$, because by supposition there is no continuous $g: [0, 1] \to R$ lying in O. Then a spherical neighborhood containing $(b, \phi(b))$ and lying in O will contain a point $(c, \phi(c))$ with $b < c$, and the function g can be continuously extended within that neighborhood to $(c, \phi(c))$. Thus, $(c, \phi(c))$ is a point of G with abscissa greater than b, and this is a contradiction. So $(b, \phi(b))$ does not belong to G.

Now consider a spherical neighborhood N' of $(b, \phi(b))$ lying in O and having radius r. Notice that every point in the *left* half of N' belongs to J, because otherwise the left half of N' would necessarily contain a point (x', y') of G and the continuous function g' associated with (x', y') could be extended continuously within O to $(b, \phi(b))$. Now, let $(a, \phi(a))$ be a point of G such that $b - r < a < b$. Now consider a neighborhood N'' of

$(a, \phi(a))$ of radius $r' < \min(b - a, a - b + r)$ such that N'' lies in G. Then, the vertical strip W of all points in $[0, 1] \times R$ with abscissa in $[a, a + r']$ is such that every vertical line lying in W contains a point of G with a positive ordinate and a point of J with a positive ordinate and therefore also a point of H with a positive ordinate. This is the point to which Vietoris arrives in his argument at the top of p. 204 of [9].

What follows is a paraphrasing of Vietoris's argument, altered slightly to serve the special purposes of this proof. Let $H_0 = H \cap W$. Let $W_1, W_2, \cdots$ be a sequence of open sets containing H_0 (W_n being the union of the $(1/2^n)$-neighborhoods of points of H_0). A sequence $n(1), n(2), \cdots$ of positive integers and a dyadic decimal $0.a_1 a_2 \cdots$ will be defined simultaneously, with M_i denoting $(a_1 + a_2 + \cdots + a_i)/i$ as the process is carried out. First pick a dyadic rational $\xi_0 = 0.a_1 \cdots a_j$ between a and $a + r'$ and then any point $c_1 = (\xi_0, \eta_0)$ of $G \cap W_1$ with $\eta_0 > 0$. Consider a square shaped neighborhood Q of c_1 which has radius $q < \eta/2$ and lies interior to $G \cap W_1 \cap W$. Now, proceed to define $a_{j+1}, a_{j+2}, \cdots$ etc., using consecutive 0's up to a stage a_k so that (1) regardless of how $0.a_1 \cdots a_j \cdots a_k \cdots$ is continued, it will differ from ξ_0 by less than q, (2) $M_k < q$ and (3) $1/k < q$. Then continue defining $a_{k+1}, a_{k+2}, \cdots$ etc., using consecutive 1's up to a stage $a_{n(1)}$ so that the point $b_1 = (\xi_1, M_{n(1)})$ is in Q (where $\xi_1 = 0.a_1 a_2 \cdots a_{n(1)}$). This can be accomplished since $|M_{i+1} - M_i| < 1/(i + 1) < q$ for each $i \geq k$. Now repeat the process, starting with any point $c_2 = (\xi_1, \eta_1)$ of $G \cap W_2$ with $\eta_1 > 0$, and then defining $a_{n(1)+1}, a_{n(2)+2}, \cdots$ etc., first using consecutive 0's and then using consecutive 1's up to a stage $a_{n(2)}$ so that $\xi_2 = 0.a_1 a_2 \cdots a_{n(2)}$ is between ξ_1 and $a + r'$ and the point $b_2 = (\xi_2, M_{n(2)})$ is in $W_2 \cap G$. Continue this process. Define $x_\omega = 0.a_1 a_2 \cdots$ and $y_\omega = \phi(x_\omega)$. Since consecutive 0's and then consecutive 1's were used in proceeding from $a_{n(r)}$ to $a_{n(r+1)}$ in the above process, it is true that for each i between $n(r)$ and $n(r + 1)$,

$$M_i \leq \max\{M_{n(r)}, M_{n(r+1)}\},$$

so that $\phi(x_\omega) = \limsup M_i = \limsup M_{n(i)}$. Therefore, the point (x_ω, y_ω) is the limit of some subsequence of $b_1, b_2, \cdots$. Also, since $b_n \in W_n$, $(x_\omega, y_\omega) \in H_0$. Then since (x_ω, y_ω) is a point of ϕ, it has a neighborhood N''' which lies in O, and N''' contains one of the points b_i. Notice that b_i lies *to the left* of (x_ω, y_ω), so that the continuous function g which lies in O, has domain an interval, and has left end $(0, 0)$ and right end b_i can be continuously extended within O to (x_ω, y_ω). So (x_ω, y_ω) belongs to G rather than H. This is a contradiction.

Remark. It is to a certain degree disappointing that ϕ satisfies (i) because it would be interesting to find a function which has a connected graph, and does not satisfy (i), but which could be described in terms of an equation (rather than by the topological techniques of [4] or the transfinite techniques of [1], [2], and [6]).

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