PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 49, Number 1, May 1975

CONVERGENCE OF AVERAGES OF POINT TRANSFORMATIONS

M. A. AKCOGLU[1] AND A. DEL JUNCO

Let $(X, \mathfrak{F}, \mu)$ be a finite measure space. An invertible transformation of X which is measure preserving in both directions is called an automorphism of X. Birkhoff's Ergodic Theorem states that if τ is an automorphism of X then the sequence

$$\frac{1}{n+1}\sum_{i=0}^{n} f(\tau^i x)$$

converges a.e. for each $f \in L_1 = L_1(X, \mathfrak{F}, \mu)$. This raises the following question. What are the necessary conditions on the matrix (a_{ni}) so that the sequence $f_n(x) = \sum_i a_{ni} f(\tau^{-i}x)$ converges a.e. for each $f \in L_1$ and for each automorphism τ? The answer is not known. Spectral considerations would suggest, however, the following conjecture. If (a_{ni}) is such that the sequence of functions $p_n(z) = \sum_i a_{ni} z^{-i}$ is uniformly bounded and pointwise convergent on the unit circle $|z| = 1$, then f_n converges a.e. In fact, recently an attempt has been made to prove this as a theorem [1]. In this note we would like to observe the following simple fact which shows that this conjecture is far from being correct.

If r is a real number, let $[r]$ denote the greatest integer which is less than or equal to r. Define a matrix (a_{ni}), $n = 1, 2, \cdots$, as

$$a_{ni} = \begin{cases} \dfrac{1}{[\sqrt{n}] + 1} & \text{if } n \le i \le [\sqrt{n}] + n, \\ 0 & \text{otherwise.} \end{cases}$$

Then the a_{ni} certainly satisfy the hypotheses of the conjecture. However, we have the following result.

Proposition. *If τ is an ergodic automorphism of a probability space $(X, \mathfrak{F}, \mu)$, then there is a set E such that there is a set B of positive measure on which $\sum_i a_{ni}\chi_E(\tau^{-i}x)$ fails to converge.*

Received by the editors July 23, 1974.
AMS (MOS) subject classifications (1970). Primary 28A65, 47A35.
[1] Research supported by NRC Grant A3974.

Proof. Let $k_m = 2^m$. By Rohlin's theorem (see e.g. [2, Theorem 8.1]) for each $m \geq 2$ there is a set $F_m \in \mathfrak{F}$ such that $\tau^i F_m$, $0 \leq i \leq k_m^2 + k_m$, are mutually disjoint and

$$\mu \bigcup_{i=0}^{k_m^2+k_m} \tau^i F_m > \frac{9}{10}.$$

Let

$$E_m = \bigcup_{i=0}^{k_m} \tau^i F_m, \qquad B_m = \bigcup_{i=2k_m}^{k_m^2+k_m} \tau^i F_m.$$

Note that

(1) $$\mu(E_m) < \frac{1}{k_m},$$

(2) $$\mu(B_m) > \left(1 - \frac{2k_m}{k_m^2 + k_m}\right)\frac{9}{10} > \frac{1}{2}.$$

Let $E = \bigcup_{m \geq 2} E_m$. Then (1) implies

(3) $$\mu(E) \leq \sum_{m \geq 2} \frac{1}{k_m} < \frac{1}{2}.$$

If $x \in \tau^{n+k_m} F_n$, $k_m \leq n \leq k_m^2$, then

(4) $$\sum_i a_{ni} \chi_{\tau^{-i} E_n}(x) = 1.$$

(4) implies that for each $x \in B_m$ there is an integer $n \geq k_m$ such that

(5) $$\sum_i a_{ni} \chi_E(\tau^{-i} x) = 1.$$

Since $\mu(B_m) > 1/2$, by Fatou's Lemma there is a set B such that $\mu(B) > 1/2$ and each $x \in B$ belongs to infinitely many of the B_m. Thus if $x \in B$, (5) holds for infinitely many integers n. However since τ is ergodic, if $\Sigma_i a_{ni} \chi_E(\tau^{-i} x)$ converges a.e., the limit function is equal to $\mu(E)$ a.e. Thus by (3) the convergence must fail.

BIBLIOGRAPHY

1. J. M. Belley, *Invertible measure preserving transformations and pointwise convergence*, Proc. Amer. Math. Soc. 43 (1974), 159–162.

2. M. Smorodinsky, *Ergodic theory, entropy*, Lecture Notes in Math., vol. 214, Springer-Verlag, New York and Berlin, 1971.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO, CANADA