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THE ZEROS OF JENSEN POLYNOMIALS ARE SIMPLE

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ABSTRACT. An entire function $f(z) = \sum_{k=0}^{\infty} a_k z^{k+m}/k!$ is said to be in the class $\mathcal{L}\text{-}\mathcal{P}$ (Laguerre-Pólya) if it can be represented in the form

$$f(z) = cz^m e^{-\alpha z^2+\beta z} \prod_n (1 - z/z_n)e^{z/z_n},$$

where $\alpha \geq 0$, c, β, and z_n are real, and $\sum_n z_n^{-2} < \infty$. A well-known result of Jensen asserts that the associated (Jensen) polynomials

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k$$

have only real zeros. Here we present an elementary proof of this fact; we also show that the zeros of $g_n(x)$ are simple.

Jensen's original proof may be found in [2]. Our proof uses the ideas of [1] and depends upon the following algebraic rule, which Pólya [3, p. 21] credits to de Gua: *A polynomial $p(x)$ with real coefficients has real, simple zeros only, if its derivatives $p'(x), p''(x), \ldots, p^{(n)}(x), \ldots$ have the property: if ξ is real and $p^{(n)}(\xi) = 0$, then $p^{(n-1)}(\xi)p^{(n+1)}(\xi) < 0$.*

We now prove this

Theorem. *Let $f(z) \in \mathcal{L}\text{-}\mathcal{P}$, $f(z) \not\equiv cz^m e^{\beta z}$. Then the zeros of the Jensen polynomials $g_n(x)$ are real and simple. Moreover, if*

$$f(z) = z^m e^{\sigma z} \prod_n (1 + z/z_n) \tag{1}$$

where $\sigma \geq 0$, $z_n > 0$ and $\sum_n z_n^{-1} < \infty$, then the zeros of $g_{n-1}(x)$ separate the zeros of $g_n(x)$.

Proof. If we set $F(z) = z^{-m}f(z)$ ($m \geq 0$ is the multiplicity of the zero of $f(z)$ at $z = 0$), then $F(z) \in \mathcal{L}\text{-}\mathcal{P}$, and a slight modification of Laguerre's inequality (see e.g., Skovgaard [5, p. 68]) shows that

$$[F^{(n)}(0)]^2 - F^{(n-1)}(0)F^{(n+1)}(0) = a_n^2 - a_{n-1}a_{n+1} > 0, \qquad n \geq 1. \tag{2}$$

Since

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$$e^z f(xz) = (xz)^m \sum_{n=0}^{\infty} g_n(x) \frac{z^n}{n!} \in \mathcal{L}\text{-}\mathcal{P}$$

for every real x, it follows from (2) that

$$\Delta_n(x) = g_n^2(x) - g_{n-1}(x) g_{n+1}(x) > 0, \quad x \neq 0, \quad n \geq 1. \tag{3}$$

Now, set

$$P_n(x) = \frac{1}{n!} x^n g_n(x^{-1}), \qquad \sigma_n(x) = \frac{n}{n+1} P_n^2(x) - P_{n-1}(x) P_{n+1}(x),$$

and observe that

$$P_n'(x) = P_{n-1}(x) \tag{4}$$

and

$$x^{2n} \Delta_n(x^{-1}) = (n-1)!(n+1)!\, \sigma_n(x). \tag{5}$$

Thus, (2) and (3), together with (5), imply

$$\sigma_n(x) > 0, \quad -\infty < x < \infty, \quad n \geq 1. \tag{6}$$

Now if $P_n^{(k)}(\xi) = 0$, then (4) and (6) applied to $\sigma_{n-k}(\xi)$ imply that $P_n^{(k-1)}(\xi) P_n^{(k+1)}(\xi) < 0$, $k = 1, \ldots, n-1$. Hence by de Gua's rule $P_n(x)$ has simple real zeros and consequently so does $g_n(x)$.

In order to prove the second assertion of the Theorem, assume that $f(z)$ is of the form (1). Then $a_k > 0$, $k = 0, 1, \ldots$, and therefore $g_n(x)$ has only negative zeros (these are simple in view of the first assertion of the Theorem). Now suppose s and t are consecutive zeros of $g_n(x)$, say $s < t$. Then the known recurrence relation [4, p. 240] $n g_n(x) = n g_{n-1}(x) + x g_n'(x)$ implies that

$$n^2 g_{n-1}(s) g_{n-1}(t) = st g_n'(s) g_n'(t) < 0.$$

Thus $g_{n-1}(x)$ vanishes in (s, t). This completes the proof of the Theorem.

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