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REGULAR FUNCTIONS $f(z)$ FOR WHICH $zf'(z)$ IS α-SPIRAL-LIKE

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ABSTRACT. Let $\mathcal{F}_\alpha^\lambda$ be the class of functions $f(z) = z + a_2 z^2 + \cdots$ which are regular in $E = \{z/|z| < 1\}$ and satisfy

$$\operatorname{Re}\{e^{i\alpha}(1 + zf''(z)/f'(z))\} > \lambda \cos \alpha$$

for some α, $|\alpha| < \pi/2$, and for some λ, $0 \le \lambda < 1$. The author finds a range on α for which $f(z)$ in $\mathcal{F}_\alpha^\lambda$ is univalent in E. In particular, the author improves upon the range on α for which $f(z) \in \mathcal{F}_\alpha^0$ is known to be univalent in E. Also a corresponding result is obtained for those functions $f(z)$ in $\mathcal{F}_\alpha^\lambda$ for which $f''(0) = 0$.

Introduction. A function $f(z) = z + a_2 z^2 + \cdots$ regular in the open unit disc $E = \{z/|z| < 1\}$ and satisfying the condition

$$\operatorname{Re}\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\} > 0, \qquad z \text{ in } E, \tag{1}$$

for some α, $|\alpha| < \pi/2$, is called an α-spiral-like function. Špaček [6] showed that such a function is univalent in E. For $\alpha = 0$, (1) defines a starlike function.

Recently Robertson [4] introduced the class of functions $f(z)$ regular in E and satisfying the condition that $zf'(z)$ is α-spiral-like in E. We denote this class of functions by $\mathcal{F}_\alpha$. It is well known that $\mathcal{F}_0$ is the class of convex functions in E. Robertson proved that $f(z) \in \mathcal{F}_\alpha$ is univalent in E if $0 < \cos\alpha \le 0.2315\ldots$. Later Libera and Ziegler [2] gave an improvement on the range of α for which $f(z)$ is univalent in E. They showed that $f(z) \in \mathcal{F}_\alpha$ is univalent in E if $0 < \cos\alpha \le 0.2564\ldots$. In this paper a slight improvement of this result is given.

Let $\mathcal{F}_\alpha^\lambda$ be the class of functions $f(z)$ which are regular in E and satisfy the conditions:

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(i) $f(0) = 0,\ f'(0) = 1$;

(ii) $f'(z) \neq 0$, z in E;

(iii) $\operatorname{Re}\{e^{i\alpha}(1 + zf''(z)/f'(z))\} > \lambda \cos\alpha$, z in E, for some α, $|\alpha| < \pi/2$ and for some λ, $0 \leq \lambda < 1$, $\mathcal{F}_\alpha^0 \equiv \mathcal{F}_\alpha$.

In §2, it is shown that $f(z) \in \mathcal{F}_\alpha^\lambda$ is univalent in E if $\lambda \geq 0.6$. For $\lambda < 0.6$, a range of α for which $f(z) \in \mathcal{F}_\alpha^\lambda$ is always univalent in E is given. It is also shown that for each α, $1/2(1 - \lambda) < \cos\alpha < 1$, there exists $f(z) \in \mathcal{F}_\alpha^\lambda$ such that $f(z)$ is not univalent in E.

In §3, we have considered those functions $f(z)$ which satisfy the condition $f''(0) = 0$. In particular, it is shown that these functions are univalent in E if $\lambda \geq 0.2$.

2. We shall need the following lemmas:

Lemma 1. *Let $P(z) = 1 + p_k z^k + \cdots$ be a regular function in E such that* $\operatorname{Re} P(z) > 0$, *$z$ in E. Let*

$$|z| = r, \quad a = \frac{1 + r^{2k}}{1 - r^{2k}}, \quad \rho = \frac{2r^k}{1 - r^{2k}}, \quad |P(z) - a| = \rho_0.$$

Then on $|z| = r$, $0 < r < 1$,

(i) $\rho_0 \leq \rho$, *and*

$$\text{(ii)} \quad |2zP'(z) - k(P^2(z) - 1)| \leq \frac{4r^{k+1}}{(1 - r^2)(1 - r^{2k})} - \frac{(1 - r^{2k})\rho_0^2}{r^{k-1}(1 - r^2)}.$$

Equality occurs in (i) and (ii) for $P(z) = (1 + z^k)/(1 - z^k)$ at $z = r$.

Inequality (i) is well known. A proof of inequality (ii) for general k is similar to the proof of (ii) for $k = 1$ as given in [1].

Lemma 2. *Let $f(z) = z + a_2 z^2 + \cdots$ be regular in E and let $w(f, z)$ denote the Schwarzian derivative of $f(z)$,*

$$w(f, z) = [f''(z)/f'(z)]' - \tfrac{1}{2}(f''(z)/f'(z))^2.$$

Then $f(z)$ is univalent in E whenever $|w(f, z)| \leq 2/(1 - r^2)^2$, z in E.

We owe this lemma to Nehari [3].

Theorem 1. *Let $f(z)$ be in $\mathcal{F}_\alpha^\lambda$, $p = (\lambda^2 \cos^2\alpha + \sin^2\alpha)^{1/2}$, and $|z| = r < 1$. Then for $p \leq \tfrac{1}{2}$,*

$$|w(f, z)| \leq \frac{2(1 - \lambda)\cos\alpha}{(1 - r^2)^2}\left\{1 + \frac{pr^2}{1 - p}\right\}, \qquad 0 \leq r < 1;$$

and for $p > \frac{1}{2}$,

$$|w(f, z)| \leq \begin{cases} \dfrac{2p(1-\lambda)\cos\alpha}{(1-r)^2}, & \dfrac{1}{p} - 1 \leq r < 1, \\[2ex] \dfrac{2(1-\lambda)\cos\alpha}{(1-r^2)^2}\left\{1 + \dfrac{pr^2}{1-p}\right\}, & 0 \leq r < \dfrac{1}{p} - 1. \end{cases}$$

$f(z)$ *is univalent in* E *for* $\lambda \geq 0.6$ *and all* α, $|\alpha| < \pi/2$. *If* $\lambda < 0.6$, $f(z)$ *is univalent in* E *for*

$$0 < \cos\alpha \leq \left\{\frac{2(1-\lambda) - ((1-\lambda)(3-5\lambda))^{\frac{1}{2}}}{4(1-\lambda)(1-\lambda^2)}\right\}^{\frac{1}{2}}.$$

If $0.5 \leq \lambda \leq 0.6$, $f(z)$ *is also univalent in* E *for*

$$\left\{\frac{2(1-\lambda) + ((1-\lambda)(3-5\lambda))^{\frac{1}{2}}}{4(1-\lambda)(1-\lambda^2)}\right\}^{\frac{1}{2}} \leq \cos\alpha \leq 1.$$

For each α, $1/2(1-\lambda) < \cos\alpha < 1$, *there exists* $f(z)$ *in* $\mathcal{F}_\alpha^\lambda$ *such that* $f(z)$ *is not univalent in* E.

Putting $\lambda = 0$ in Theorem 1, we arrive at the following result which is an improvement of earlier results due to Robertson [4] and Libera and Ziegler [2].

Corollary 1. *If* $f(z)$ *is in* $\mathcal{F}_\alpha$ *and* $|z| = r < 1$, *then* $f(z)$ *is univalent in* E *for* $0 < \cos\alpha \leq (\sqrt{6} - \sqrt{2})/4 = .2588\cdots$.

The above result is the best possible one obtainable solely from an application of Nehari's test for univalence (the only one used in this situation by the previous authors).

Proof of Theorem 1. Let $f(z)$ be in $\mathcal{F}_\alpha^\lambda$. Then there exists a function $P(z)$, Re $P(z) > 0$ for z in E, such that

$$e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \cos\alpha\{(1-\lambda)P(z) + \lambda\} + i\sin\alpha.$$

On simplification this gives

$$f''(z)/f'(z) = (1-\lambda)\cos\alpha\, e^{-i\alpha}(P(z) - 1)/z,$$

and

$$(f''(z)/f'(z))' = (1-\lambda)\cos\alpha\, e^{-i\alpha}(zP'(z) + 1 - P(z))/z^2.$$

Consequently

$$w(f, z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

$$= \frac{(1-\lambda)\cos\alpha\, e^{-i\alpha}}{2z^2}[2zP'(z) + 1 - P^2(z) + (\lambda\cos\alpha + i\sin\alpha)e^{-i\alpha}(P(z)-1)^2].$$

Therefore

(1) $$|w(f, z)| \leq \frac{(1-\lambda)\cos\alpha}{2r^2}[|2zP'(z) + 1 - P^2(z)| + p|P(z)-1|^2].$$

From Lemma 1, we have

(2) $$|2zP'(z) + 1 - P^2(z)| \leq \rho^2 - \rho_0^2$$

where

$$\rho_0 = |P(z) - a| \leq \rho, \qquad a = \frac{1+r^2}{1-r^2}, \qquad \rho = \frac{2r}{1-r^2}.$$

Also we have

$$|P(z) - 1| = |P(z) - a + a - 1| \leq |P(z) - a| + \rho r,$$

i.e.,

(3) $$|P(z) - 1| \leq \rho_0 + \rho r.$$

Using (2) and (3) in (1), we have

(4) $$|w(f, z)| \leq \frac{(1-\lambda)\cos\alpha}{2r^2} x(\rho_0),$$

where

$$x(\rho_0) = \rho^2 - \rho_0^2 + p(\rho_0 + \rho r)^2.$$

Now $x'(\rho_0) = 0$ at $\rho_0 = \rho_1$, $\rho_1 = \rho r p/(1-p)$. Also $x''(\rho_0) < 0$ always. Therefore $x(\rho_0)$ attains its maximum value at $\rho_0 = \rho_1$ provided $\rho_1 < \rho$. Otherwise $x(\rho_0)$ attains its maximum value at $\rho_0 = \rho$. Now the condition $\rho_1 < \rho$ is satisfied for all r, $0 \leq r < 1$, if $p \leq \frac{1}{2}$. When $p > \frac{1}{2}$, the condition $\rho_1 < \rho$ is satisfied for r in the interval $0 \leq r < 1/p - 1$. Thus we have

(5a) $$x(\rho_0) \leq \frac{4r^2}{(1-r^2)^2}\left\{1 + \frac{pr^2}{1-p}\right\}, \qquad 0 \leq r < 1,$$

if $p \leq \frac{1}{2}$; and

$$\text{(5b)} \qquad \varkappa(\rho_0) \leq \begin{cases} \dfrac{4r^2}{(1-r^2)^2}\left\{1 + \dfrac{pr^2}{1-p}\right\}, & 0 \leq r < \dfrac{1}{p} - 1, \\ \dfrac{2p(1-\lambda)\cos\alpha}{(1-r)^2}, & \dfrac{1}{p} - 1 \leq r < 1, \end{cases}$$

if $p > \frac{1}{2}$. In conjunction with (5a) and (5b), (4) gives upper bounds on $|w(f, z)|$ as given in the statement of the theorem.

Now $|w(f, z)| \leq 2/(1-r^2)^2$ leads to the following inequalities:

$$\text{(6)} \qquad (1-\lambda)\cos\alpha\left\{1 + \frac{pr^2}{1-p}\right\} \leq 1 \quad \text{for } 0 \leq r < \frac{1}{p} - 1;$$

and

$$\text{(7)} \qquad p(1-\lambda)(1+r)^2\cos\alpha \leq 1 \quad \text{for } 1/p - 1 \leq r < 1.$$

Now (6) is satisfied provided $(1-\lambda)\cos\alpha(1+r) \leq 1$, i.e., if

$$\text{(8)} \qquad (1-\lambda)\cos\alpha \leq p = (\lambda^2\cos^2\alpha + \sin^2\alpha)^{\frac{1}{2}}.$$

The above inequality is satisfied for all α, $|\alpha| < \pi/2$, if $\lambda \geq 0.5$. In the case $\lambda < 0.5$, (8) is satisfied when

$$\text{(9)} \qquad 0 < \cos\alpha \leq 1/\sqrt{2(1-\lambda)}.$$

Now (7) is satisfied provided

$$\text{(10)} \qquad 1 - 16(1-\lambda)^2\cos^2\alpha + 16(1-\lambda^2)(1-\lambda)^2\cos^4\alpha \geq 0.$$

(10) is true for all α, $|\alpha| < \pi/2$, if $\lambda \geq 0.6$. In the case $\lambda < 0.6$, (10) is satisfied when

$$0 < \cos\alpha \leq \left\{\frac{2(1-\lambda) - ((1-\lambda)(3-5\lambda))^{\frac{1}{2}}}{4(1-\lambda)(1-\lambda^2)}\right\}^{\frac{1}{2}}.$$

For $0.5 \leq \lambda < 0.6$, (10) is satisfied also when

$$\left\{\frac{2(1-\lambda) + ((1-\lambda)(3-5\lambda))^{\frac{1}{2}}}{4(1-\lambda)(1-\lambda^2)}\right\}^{\frac{1}{2}} \leq \cos\alpha < 1.$$

The condition $p > \frac{1}{2}$ is equivalent to the condition

$$\cos\alpha < \sqrt{3}/2\sqrt{1-\lambda^2}$$

(< 1 for $\lambda < 0.5$). This is no restriction on α if $\lambda \geq 0.5$. If $\lambda < 0.5$, then

$$\frac{\sqrt{3}}{2\sqrt{1-\lambda^2}} > \frac{1}{\sqrt{2(1-\lambda)}} > \left\{\frac{2(1-\lambda)-((1-\lambda)(3-5\lambda))^{1/2}}{4(1-\lambda)(1-\lambda^2)}\right\}^{1/2}.$$

From the above results and Lemma 2, we conclude that

(i) For $\lambda \geq 0.6$, $f(z)$ is univalent in E for all α, $|\alpha| < \pi/2$.

(ii) For $\lambda < 0.6$, $f(z)$ is univalent in E when

$$0 < \cos\alpha \leq \left\{\frac{2(1-\lambda)-((1-\lambda)(3-5\lambda))^{1/2}}{4(1-\lambda)(1-\lambda^2)}\right\}^{1/2}.$$

(iii) For $0.5 \leq \lambda < 0.6$, $f(z)$ is univalent in E also when

$$\left\{\frac{2(1-\lambda)+((1-\lambda)(3-5\lambda))^{1/2}}{4(1-\lambda)(1-\lambda^2)}\right\}^{1/2} \leq \cos\alpha \leq 1.$$

This completes the proof of the first part of the theorem.

It may be mentioned here that the upper bound on $|w(f, z)|$ in the case $p > 1/2$ is sharp for $1/p - 1 \leq r < 1$. Equality is attained for

$$f(z) = \frac{1}{\mu}\left[\frac{1}{(1-z)^{\mu}} - 1\right], \qquad \mu = 2(1-\lambda)\cos\alpha e^{-i\alpha} - 1,$$

at $z = r$. It is easy to see that these functions are in $\mathcal{F}_\alpha^\lambda$.

To prove the second part of the theorem, we consider the functions

$$f(z) = \frac{1}{\mu}\left[\frac{1}{(1-z)^{\mu}} - 1\right] \tag{11}$$

where $\mu + 1 = |\mu + 1|e^{-i\alpha}$ and

$$|\mu + \lambda\cos\alpha e^{-i\alpha}| \leq ((1-\lambda)^2\cos^2\alpha + \sin^2\alpha)^{1/2}.$$

It is easy to verify that $f(z)$ is in $\mathcal{F}_\alpha^\lambda$. Royster [5] has proved that the necessary and sufficient condition for the functions (11) to be univalent is that either $|\mu + 1| \leq 1$ or $|\mu - 1| \leq 1$. We conclude that $f(z)$ is in $\mathcal{F}_\alpha^\lambda$ and is not univalent in E when μ lies in the set D defined by the inequalities

$$|\mu + \lambda\cos\alpha e^{-i\alpha}| \leq \{(1-\lambda)^2\cos^2\alpha + \sin^2\alpha\}^{1/2}, \qquad |\mu + 1| > 1, |\mu - 1| > 1.$$

It is easy to prove that for each α, for which $1/2(1-\lambda) < \cos\alpha < 1$, we can choose a number μ so that μ lies in the set D. The function $f(z)$ then belongs to $\mathcal{F}_\alpha^\lambda$ but is not univalent in E. Similarly, when $0 < \cos\alpha \leq 1/2(1-\lambda)$, we can choose a number μ so that $|\mu + 1| \leq 1$,

$$|\mu + \lambda \cos \alpha e^{-i\alpha}| \leq \{(1-\lambda)^2 \cos^2 \alpha + \sin^2 \alpha\}^{1/2}$$

and $\mu + 1 = |\mu + 1| e^{-i\alpha}$, and then $f(z)$ belongs to $\mathcal{F}_\alpha^\lambda$ and is univalent in E.

We now state without proof the sharp radius of convexity of the class $\mathcal{F}_\alpha^\lambda$.

Theorem 2. *Let $f(z)$ be in $\mathcal{F}_\alpha^\lambda$. Then $f(z)$ is convex in*

$$|z| < \frac{(1-\lambda)\cos\alpha - \{1 - (1-\lambda^2)\cos^2\alpha\}^{1/2}}{2(1-\lambda)\cos^2\alpha - 1}.$$

This result is sharp for

$$f(z) = \frac{1}{\mu}\left[\frac{1}{(1-z)^\mu} - 1\right], \qquad \mu = -1 + 2(1-\lambda)\cos\alpha e^{-i\alpha}.$$

3. In this section we shall consider those functions $f(z)$ in $\mathcal{F}_\alpha^\lambda$ for which $f''(0) = 0$.

Theorem 3. *Let $f(z)$ be in $\mathcal{F}_\alpha^\lambda$ and $f''(0) = 0$. Then, for $|z| = r < 1$, we have*

$$|w(f, z)| \leq \frac{2(1-\lambda)\cos\alpha}{(1+r^2)(1-r^2)^2} [r^2\{(1+\lambda)^2 + (1-\lambda)(3+\lambda)\sin^2\alpha\}^{1/2}$$

$$+ \{(1+\lambda r^4)^2 \cos^2\alpha + (1+r^4)^2 \sin^2\alpha\}^{1/2}]$$

and $f(z)$ is univalent in E for $0.2 \leq \lambda < 1$. For $\lambda < 0.2$, $f(z)$ is univalent in E for those α which satisfy either

$$0 < \cos\alpha \leq \left\{\frac{2(1-\lambda) - ((1-\lambda)(1-5\lambda))^{1/2}}{(1-\lambda)^2(3+\lambda)}\right\}^{1/2}$$

or

$$\left\{\frac{2(1-\lambda) + (1-\lambda)(1-5\lambda)^{1/2}}{(1-\lambda)^2(3+\lambda)}\right\}^{1/2} \leq \cos\alpha \leq 1.$$

Putting $\lambda = 0$ in Theorem 3, we arrive at

Corollary 3.1. *Let $f(z)$ be in $\mathcal{F}_\alpha$ and $f''(0) = 0$. Then $f(z)$ is univalent in E when either $\alpha = 0$, or $0 < \cos\alpha \leq 1/\sqrt{3} = 0.5773\cdots$.*

Proof of Theorem 3. Proceeding as in Theorem 1, we have

$$w(f, z) = \frac{(1-\lambda)\cos\alpha e^{-i\alpha}}{2z^2}\,[\{2zP'(z) - 2(P^2(z)-1)\} + (P(z)-1)$$
$$\cdot\{P(z) + 1 + (\lambda\cos\alpha + i\sin\alpha)e^{-i\alpha}(P(z)-1)\}]$$

where $P(z) = 1 + p_2z^2 + \cdots$ satisfies Re $P(z) > 0$ for z in E. Therefore

(1)
$$|w(f, z)| \le \frac{(1-\lambda)\cos\alpha}{2r^2}\,[|2zP'(z) - 2(P^2(z)-1)| + |P(z)-1|$$
$$\cdot\,|P(z) + 1 + (\lambda\cos\alpha + i\sin\alpha)e^{-i\alpha}(P(z)-1)|].$$

From Lemma 2, we have

(2)
$$|2zP'(z) - 2(P^2(z)-1)| \le \frac{4r^3}{(1-r^2)(1-r^4)} - \frac{(1+r^2)}{r}\rho_0^2$$

where

$$\rho_0 = |P(z) - a|, \qquad a = \frac{1+r^4}{1-r^4}, \qquad \rho = \frac{2r^2}{1-r^2}, \qquad \rho_0 \le \rho.$$

Also we have

(3)
$$|P(z) - 1| = |P(z) - a + \rho r^2| \le \rho_0 + \rho r^2,$$

and

$$|P(z) + 1 + (\lambda\cos\alpha + i\sin\alpha)e^{-i\alpha}(P(z)-1)|$$
$$= |(P(z)-a)(1 + (\lambda\cos\alpha + i\sin\alpha)e^{-i\alpha})$$

(4)
$$+ ((a+1) + (\lambda\cos\alpha + i\sin\alpha)e^{-i\alpha}(a-1))|$$
$$\le \rho_0\{(1+\lambda)^2 + (1-\lambda)(3+\lambda)\sin^2\alpha\}^{1/2}$$
$$+ \rho\left\{\left(\frac{1}{r^2} + \lambda r^2\right)^2\cos^2\alpha + \left(\frac{1}{r^2} + r^2\right)^2\sin^2\alpha\right\}^{1/2}.$$

Using (2), (3) and (4) in (1), we have

$$|w(f, z)| \le \frac{(1-\lambda)\cos\alpha}{2r^2}\,x(\rho_0)$$

where

$$x(\rho_0) = \frac{4r^3}{(1-r^2)(1-r^4)} - \frac{(1-r^4)}{r(1-r^2)}\rho_0^2$$
$$+ (\rho_0 + \rho r)^2 \Bigg[\rho_0\{(1+\lambda)^2 + (1-\lambda)(3+\lambda)\sin^2\alpha\}^{1/2}$$
$$+ \rho\left\{\left(\frac{1}{r^2} + \lambda r^2\right)^2 \cos^2\alpha + \left(\frac{1}{r^2} + r^2\right)^2 \sin^2\alpha\right\}^{1/2}\Bigg].$$

Now $x'(\rho_0) > 0$ for $0 \leq \rho_0 < \rho_1$, where

$$\rho_1 = \frac{\rho[\{(1+\lambda r^4)^2\cos^2\alpha + (1+r^4)^2\sin^2\alpha\}^{1/2} + r^4\{(1+\lambda)^2 + (1-\lambda)(3+\lambda)\sin^2\alpha\}^{1/2}]}{2r[1 + r^2 - r\{(1+\lambda)^2 + (1-\lambda)(3+\lambda)\sin^2\alpha\}^{1/2}]}.$$

But $\rho_0 \leq \rho$ and it is easy to verify that $\rho \leq \rho_1$. Thus we conclude that $x(\rho_0)$ increases monotonically as ρ_0 increases from 0 to ρ and $x(\rho_0)$ assumes its maximum value at $\rho_0 = \rho$. Thus we have

$$|w(f,z)| \leq \frac{2(1-\lambda)\cos\alpha}{(1+r^2)(1-r^2)^2}[r^2\{(1+\lambda)^2 + (1-\lambda)(3+\lambda)\sin^2\alpha\}^{1/2}$$
$$+ \{(1+\lambda r^4)^2\cos^2\alpha + (1+r^4)^2\sin^2\alpha\}^{1/2}]$$

$$\leq \frac{2(1-\lambda)\cos\alpha}{(1+r^2)(1-r^2)^2}[r^2\{(1+\lambda)^2 + (1-\lambda)(3+\lambda)\sin^2\alpha\}^{1/2}$$
$$+ \{(1+\lambda)^2\cos^2\alpha + 4\sin^2\alpha\}^{1/2}]$$

$$= \frac{2(1-\lambda)\cos\alpha}{(1-r^2)^2}\{(1+\lambda)^2 + (1-\lambda)(3+\lambda)\sin^2\alpha\}^{1/2}.$$

From Lemma 2, it follows that $f(z)$ is univalent in E for those values of α for which

$$(1-\lambda)\cos\alpha\{(1+\lambda)^2 + (1-\lambda)(3+\lambda)\sin^2\alpha\}^{1/2} \leq 1.$$

This last inequality is equivalent to

(5) $$1 - 4(1-\lambda)^2\cos^2\alpha + (1-\lambda)^3(3+\lambda)\cos^4\alpha \geq 0$$

which is true for all values of $\cos\alpha$ provided

$$16(1-\lambda)^4 - 4(1-\lambda)^3(3+\lambda) \leq 0,$$

i.e., if $\lambda \geq 0.2$.

When $\lambda < 0.2$, (5) is satisfied for those values of α for which either

$$0 < \cos\alpha \le \left\{ \frac{2(1-\lambda) - ((1-\lambda)(1-5\lambda))^{1/2}}{(1-\lambda)^2(3+\lambda)} \right\}^{1/2},$$

or

$$\left\{ \frac{2(1-\lambda) + ((1-\lambda)(1-5\lambda))^{1/2}}{(1-\lambda)^2(3+\lambda)} \right\}^{1/2} \le \cos\alpha \le 1.$$

Theorem 4. *Let* $f(z)$ *be in* $\mathcal{F}_\alpha^\lambda$ *and* $f''(0) = 0$. *Then* $f(z)$ *is convex in*

$$|z| < \left\{ \frac{(1-\lambda)\cos\alpha - \{1 - (1-\lambda^2)\cos^2\alpha\}^{1/2}}{2(1-\lambda)\cos^2\alpha - 1} \right\}^{1/2}.$$

This result is sharp, the extremal function being

$$f(z) = \int_0^z \frac{dz}{(1-z^2)^\mu}, \qquad \mu = (1-\lambda)\cos\alpha e^{-i\alpha}.$$

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REFERENCES

1. R. S. Gupta, *Radius of univalence and starlikeness of a class of analytic functions*, J. Austral. Math. Soc. 14 (1972), 1–8. MR 47 #5239.

2. R. J. Libera and M. R. Ziegler, *Regular functions* $f(z)$ *for which* $zf'(z)$ *is* α*-spiral*, Trans. Amer. Math. Soc. 166 (1972), 361–370.

3. Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. 55 (1949), 545–551. MR 10, 696.

4. M. S. Robertson, *Univalent functions* $f(z)$ *for which* $zf'(z)$ *is spirallike*, Michigan Math. J. 16 (1969), 97–101. MR 39 #5785.

5. W. C. Royster, *On the univalence of a certain integral*, Michigan Math. J. 12 (1965), 385–387. MR 32 #1342.

6. L. Špaček, *Přispěvek k teorii funkci prostyčh*, Časopis Pěst. Mat. Fys. 62 (1933), 12–19.

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