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UNIQUE BEST NONLINEAR APPROXIMATION IN HILBERT SPACES

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ABSTRACT. Using the notion of curvature of a manifold, developed by J. R. Rice and recently studied by E. R. Rozema and the second named author, the authors prove the following result: Let H be a Hilbert space and F map R^n into H such that F is a homeomorphism onto $\mathfrak{F} = F(R^n)$ and is twice continuously Fréchet differentiable. Then if $F'(\alpha) \cdot R^n$ is of dimension n for all $\alpha \in R^n$, the manifold $\mathfrak{F}$ has finite curvature everywhere. It follows that there is a neighborhood $\mathfrak{U}$ of $\mathfrak{F}$ such that each $u \in \mathfrak{U}$ has a *unique* best approximation from $\mathfrak{F}$. However, these results do not hold in general for uniformly smooth Banach spaces.

1. **Introduction.** Nonlinear approximation arises naturally in many problems. In particular, where algorithmic solutions are desired, uniqueness questions are of great concern. Many recent papers have been written on this subject; for example, see [3], [4], [5], [6], and [7]. In [4], Rozema and the second named author have studied the question of unique best approximation in uniformly smooth Banach spaces. This note is a completion of [4] in the sense that for a Hilbert space it gives a sufficient condition for finite curvature, and hence unique best approximation, in terms of smoothness of the map.

Let F map R^n into a Hilbert space H. We will denote by $\mathfrak{F}$ the image of R^n under F and by $F'(\alpha)$ the Fréchet derivative of F at $\alpha \in R^n$. Throughout this paper we will assume that F satisfies the following three conditions:

(a) F is a homeomorphism onto its image $\mathfrak{F}$ (where $\mathfrak{F}$ is given the topology induced by H).

(b) The first and second Fréchet derivatives of F exist and are continuous. (We will denote the second Fréchet derivative of F at α by $F''(\alpha)$.)

(c) The dimension of $F'(\alpha) \cdot R^n$ is n for each $\alpha \in R^n$.

As in [4], we define the tangent plane $T(x)$ to $\mathfrak{F}$ at $x = F(\alpha)$ by $T(x) = x + F'(\alpha) \cdot R^n$, and the set of normals $\mathfrak{N}(x)$ to $\mathfrak{F}$ at x by $\mathfrak{N}(x) =$

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$\{y: (y - x) \perp T(x)\}$. Let x and z be in $\mathcal{F}$, and for $y \in \mathfrak{N}(x)$, we set

$$\rho(x, y, z) = \inf \|y(z) - z\|$$

where the infimum is taken over all the $y(z)$ that lie on the line generated by x and y and satisfy $\|y(z) - x\| = \|y(z) - z\|$. If no $y(z)$ exists, we set $\rho(x, y, z) = \infty$. Let

$$\rho(x, y) = \liminf_{z \to x,\, z \in \mathcal{F}} \rho(x, y, z).$$

The radius of curvature $\rho(x)$ of $\mathcal{F}$ at x is defined in [4] as

$$\rho(x) = \inf_{y \in \mathfrak{N}(x)} \rho(x, y)$$

and the curvature $\sigma(x)$ of $\mathcal{F}$ at x is defined by $\sigma(x) = 1/\rho(x)$. For simplicity we will only consider real Hilbert spaces.

2. Finite curvature in terms of smoothness.

We first establish the following theorem for a Hilbert space H. In the next section we will see that this result does not necessarily hold for a uniformly smooth Banach space in general.

Theorem 1. *Let* $F: R^n \to H$ *satisfy* (a), (b), *and* (c). *Then the manifold* $\mathcal{F}$ *has finite curvature everywhere.*

Proof. We first calculate $\rho(x, y, z)$ where $x = F(\alpha)$ and $z = F(\beta)$ with $\alpha, \beta \in R^n$, and y lies on $\mathfrak{N}(x)$. We pick the element $y(z) \in \mathfrak{N}(x)$ where $y(z) = x + t(y - x)$, $t \in R^1$, so that $\|y(z) - x\| = \|y(z) - z\|$. Hence,

$$\|x\|^2 - 2\langle x + t(y - x), x\rangle = \|z\|^2 - 2\langle x + t(y - x), z\rangle.$$

Solving for t, we obtain

$$t = \frac{\|z\|^2 + \|x\|^2 - 2\langle x, z\rangle}{2\langle y - x, z - x\rangle} = \frac{\|z - x\|^2}{2\langle y - x, z - x\rangle},$$

and

$$\rho(x, y, z) = \|y(z) - z\| = \|y(z) - x\| = \frac{\|z - x\|^2 \|y - x\|}{2|\langle y - x, z - x\rangle|}.$$

Fix $x \in \mathcal{F}$. To show that $\mathcal{F}$ has finite curvature at x, we will bound $\rho(x, y, z)$ from below. Subtracting and adding $\langle y - x, F'(\alpha) \cdot (\beta - \alpha)\rangle$ in the denominator of $\rho(x, y, z)$ yields

$$\langle y - x, z - x\rangle = \langle y - x, (z - x) - F'(\alpha) \cdot (\beta - \alpha)\rangle$$

since y is in $\mathfrak{N}(x)$. Using the Schwarz inequality, we have

$$|\langle y - x, z - x\rangle| \leq \|y - x\| \, \|(z - x) - F'(\alpha) \cdot (\beta - \alpha)\|.$$

Hence,

$$\rho(x, y, z) \geq \frac{\|z - x\|^2}{2\|(z - x) - F'(\alpha) \cdot (\beta - \alpha)\|}.$$

Next, we want to find an upper bound for the above denominator when z is close to x. Since F is twice continuously Fréchet differentiable in a neighborhood of α, we have (cf. [2, pp. 99, 180])

$$\|F(\beta) - F(\alpha) - F'(\alpha) \cdot (\beta - \alpha)\|$$

$$= \left\| \left[\int_0^1 (1 - s)F''(\alpha + s(\beta - \alpha))\, ds \right] \cdot (\beta - \alpha)^{(2)} \right\| \leq c_1 |\beta - \alpha|^2$$

for some positive constant c_1 independent of β if β is close to α. Here, $(\beta - \alpha)^{(2)}$ stands for $((\beta - \alpha), (\beta - \alpha))$ and $|\cdot|$ is a norm in R^n. This gives, for β close to α (or z close to x),

$$\rho(x, y, z) \geq \frac{\|F(\beta) - F(\alpha)\|^2}{2c_1 |\beta - \alpha|^2}.$$

It is known (cf. [1, p. 219], [4]) that

$$F(\beta) - F(\alpha) = \lim_{i \to \infty} \left(\sum_{j=1}^{i} t_{ij} F'(\gamma_{ij}) \right) \cdot (\beta - \alpha)$$

when $t_{ji} \geq 0$, $\Sigma_{j=1}^{i} t_{ij} = 1$, and γ_{ij} lies on the line segment formed by α and β, $1 \leq j \leq i$, $i = 1, 2, \cdots$. By assumption (c), we see that there is a positive constant c_2 independent of β so that

$$\|F'(\alpha) \cdot (\beta - \alpha)\| \geq c_2 |\beta - \alpha|$$

for all β. From the continuity of F', it follows that when β is close to α,

$$\left\| \lim_{i \to \infty} \sum_{j=1}^{i} t_{ij} F'(\gamma_{ij}) - F'(\alpha) \right\| \leq c_2/2.$$

Thus, for β close to α,

$$- \left\| \lim_{i \to \infty} \left\{ \sum_{j=1}^{i} t_{ij} F'(\gamma_{ij}) \right\} \cdot (\beta - \alpha) \right\| + \|F'(\alpha) \cdot (\beta - \alpha)\| \leq \tfrac{1}{2} c_2 |\beta - \alpha|,$$

or

$$\tfrac{1}{2}c_2|\beta - \alpha| \le \lim_{i\to\infty} \left\| \sum_{j=1}^{i} t_{ij}F'(\gamma_{ij}) \cdot (\beta - \alpha) \right\| = \|F(\beta) - F(\alpha)\|,$$

so that

$$\rho(x, y, z) \ge \frac{c_2|\beta - \alpha|^2}{2c_1|\beta - \alpha|^2} = c_3 > 0$$

for all $z = F(\beta)$ close to $x = F(\alpha)$. Hence,

$$\rho(x, y) = \liminf_{z\to x,\, z\in\mathfrak{F}} \rho(x, y, z) \ge c_3$$

and $\rho(x) = \inf_{y\in\mathfrak{N}(x)}\rho(x, y) \ge c_3$, or $\sigma(x) \le 1/c_3 < \infty$. This completes the proof of the theorem.

3. Unique and nonunique best approximation, and an application. As an immediate consequence of Theorem 1 above and Theorem 4.1 in [4], we have the following:

Theorem 2. *Let* $F: R^n \to H$ *satisfy* (a), (b), *and* (c). *There is a neighborhood* $\mathfrak{U}$ *of* $\mathfrak{F}$ *such that each* $u \in \mathfrak{U}$ *has a unique best approximation from* $\mathfrak{F}$.

Theorems 1 and 2 could have been stated in terms of "pieces" of manifolds in the following way. Let Ω be an open subset of R^n and let $F: \Omega \to H$ satisfy (a), (b), and (c) with the exception that (c) is modified to (c′) by changing "for all $\alpha \in R^n$" to "for all $\alpha \in \Omega$". In particular, we have the following

Corollary. *Let* Ω *be an open subset of* R^n *and let* $F: \Omega \to H$ *satisfy* (a), (b), *and* (c′). *Then* $F(\Omega) = \mathfrak{F}$ *has a neighborhood* $\mathfrak{U}$ *so that every* $u \in \mathfrak{U}$ *has a unique best approximation in* $\mathfrak{F}$.

The proof of this corollary requires no new idea and will be omitted. As an example, consider the spline mapping $F: R^{2N+n+1} \to L^2[0, 1]$ defined by

$$F(\alpha) = F(\alpha_0, \cdots, \alpha_{2N+n}) = \sum_{j=0}^{n} \alpha_j t^j + \sum_{j=1}^{N} \alpha_{n+j}(t - \alpha_{N+n+j})_+^n$$

where for a real number t, $(t)_+ = \max(t, 0)$. Let $\Omega_1 = \{\alpha \in R^{2N+n+1}: 0 < \alpha_{n+N+1} < \cdots < \alpha_{n+2N} < 1,\ \alpha_j \ne 0$ for $n + 1 \le j \le n + N\}$ and $\Omega = \{\alpha \in \Omega_1:$

$|\alpha| < r < \infty\}$. Clearly, $F: \Omega \rightarrow L^2[0, 1]$ satisfies (a), (b), and (c$'$) if $n \geq 3$. Indeed, by considering $\alpha \in \Omega_1$, we can guarantee that (c$'$) holds, and in order to guarantee that (a) holds, we must bound $\alpha \in \Omega_1$, and hence, we restrict ourselves to Ω. Finally, for $n \geq 3$, it is clear that (b) holds. Thus the corollary applies and we may conclude that $\mathfrak{F} = F(\Omega)$ has a neighborhood $\mathfrak{U}$ of uniqueness. That is, every $L^2[0, 1]$ function in $\mathfrak{U}$ has a unique spline approximation in $L^2[0, 1]$ from $\mathfrak{F}$. The reader can see from the method of proof that in this example the constants c_1 and c_2, which determine $\mathfrak{U}$, can be estimated.

These results are in some sense surprising since they are not true in general when the range of F is just a uniformly smooth Banach space. For example, let $l^p(2)$, $1 \leq p < \infty$, be the two-dimensional space R^2 with the norm $\|(x, y)\| = (|x|^p + |y|^p)^{1/p}$ and let $F: R^1 \rightarrow l^p(2)$ satisfy (a), (b), and (c) with $F(t) = (\cos t, \sin t)$ in a neighborhood Ω of $t = 0$. Then F is clearly infinitely Fréchet differentiable in Ω. If $p > 2$ and $\epsilon > 0$ is small enough, it is easy to see that the points $(s, 0)$ with $1 - \epsilon < s < 1$ do not have unique best approximations from $\mathfrak{F} = F(\Omega)$. Since finite curvature implies the existence of a neighborhood of uniqueness, we can conclude that $\mathfrak{F}$ has infinite curvature at $t = 0$.

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