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A TENSOR APPROACH TO INTERPOLATION PHENOMENA IN DISCRETE ABELIAN GROUPS

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ABSTRACT. We show that a tensor algebra setting is an efficient tool that separates Sidonicity from other interpolation properties.

In this note we show that a tensor algebra setting [11] is a convenient framework where Sidonicity can be efficiently separated from other interpolation properties. Non-Sidon $\Lambda(p)$-sets in any Γ (see A below) were first obtained in [3] and [5]; our proof, similar to the one in [3, Théorème 5, p. 359] is more direct. The Rosenthal property (B) of some non-Sidon sets was first displayed in [9], where an appeal was made to the notion of "sup-norm partitions" (see also [1] and [2]). C was first observed in [4].

In what follows below, Γ is a discrete abelian group, and $\hat{\Gamma} = G$. Without loss of generality, we assume that Γ is a countable group. We refer to [11] for standard notation and facts. Let $E \subset \Gamma$; we set

$$A(E) = L^1(G)\hat{}/\{f \in L^1(G): \hat{f} = 0 \text{ on } E\},$$

and

$$B(E) = M(G)\hat{}/\{\mu \in M(G): \hat{\mu} = 0 \text{ on } E\}.$$

If $K(G)$ is a subspace of $L^1(G)$, we set

$$K_E(G) = \{f \in K(G): \hat{f} = 0 \text{ off } E\}.$$

Definitions. (a) $E \subset \Gamma$ is a Sidon set if $L^\infty_E(G) = A_E(G)$ $(= l^1(E)\hat{})$; equivalently, $A(E) = c_0(E)$.

(b) Let $1 < p < \infty$. $E \subset \Gamma$ is a $\Lambda(p)$-set if $L^1_E(G) = L^p_E(G)$.

(c) E is a Rosenthal set if $L^\infty_E(G) = C_E(G)$.

(d) Let $1 \leq p < 2$. E is a p-Sidon set if $C_E(G)\hat{} \subset l^p(E)$.

If E is a Sidon set, then E is a Rosenthal set, a p-Sidon set, and a $\Lambda(p)$-set, for all p. The first two claims are trivial to verify, whereas the third is not (see 5.7.7 of [11], and A below).

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Definition. We say that $S \subset \Gamma$ is a dissociate set if whenever the equality $\Sigma_{j=1}^{N} \omega_j \alpha_j = 0$, where $\omega_j = -1, 0, 1$, and $\{\alpha_j\}_{j=1}^{N} \subset S$, implies that $\omega_j = 0$ for all j.

It is well known that every infinite discrete abelian group contains a dissociate set.

Definition. Let S_1 and S_2 be any countably infinite sets. We set

$$l^{\infty}(S_1) \hat{\otimes} l^{\infty}(S_2) = \Big\{\phi \in l^{\infty}(S_1 \times S_2): \phi = \sum f_j g_j, \text{ where } f_j \in l^{\infty}(S_1),$$
$$g_j \in l^{\infty}(S_2), \quad \text{and} \quad \sum \|f_j\|_{\infty} \|g_j\|_{\infty} < \infty\Big\}.$$

We define $c_0(S_1) \hat{\otimes} c_0(S_2)$ similarly, and for $\phi \in l^{\infty}(S_1 \times S_2)$ or $c_0(S_1 \times S_2)$ we set $\|\phi\|_{\otimes} = \inf \{\Sigma \|f_j\|_{\infty} \|g_j\|_{\infty}: \phi = \Sigma f_j g_j\}$. We refer to [12] for a study of the above tensor algebras. We shall need the following:

Fact V (Theorem 3.1 of [11]). Let E and F be infinite disjoint subsets of Γ so that $E \cup F$ is dissociate. Then $A(E + F) \approx c_0(E) \hat{\otimes} c_0(F)$, and $l^{\infty}(E) \hat{\otimes} l^{\infty}(F)$ is a closed subalgebra of $B(E + F)$. (Note that we can freely associate functions in $l^{\infty}(E + F)$ with functions in $l^{\infty}(E \times F)$.)

By the above, since $c_0(E) \hat{\otimes} c_0(F) \subsetneq c_0(E + F)$ (see, for example, Lemma VIII.10.5 of [6]), $E + F$ is not a Sidon set.

In what follows below, $E = \{\lambda_j\}_{j=1}^{\infty}$ and $F = \{\nu_j\}_{j=1}^{\infty}$ are infinite disjoint subsets of Γ, and $E \cup F$ is a dissociate set.

A. *$E + F$ ($E + E$) is a $\Lambda(p)$-set for all p.* The proof that every Sidon set in Γ is a $\Lambda(p)$-set for all p (see 5.7.7 in [11]) is based on the prior knowledge that an infinite independent set in $\bigoplus Z_2$ is a $\Lambda(p)$-set for all p. We follow a similar route by making use of

Lemma. *There exists Γ_1, an infinite discrete abelian group ($\hat{\Gamma}_1 = H$), and $S_1 = \{\alpha_j\}_{j=1}^{\infty}$, $S_2 = \{\beta_j\}_{j=1}^{\infty}$, two infinite disjoint subsets of Γ_1, so that $S_1 \cup S_2$ is dissociate and $S_1 + S_2$ is a $\Lambda(p)$-set for all p.*

See [9], for example.

Now, let $p > 2$ and let f be any trigonometric polynomial in $L^1_{E+F}(G)$:

$$f(g) = \sum a_{ij} (\lambda_i + \nu_j, g).$$

We use S_1 and S_2 in Γ_1 of the above Lemma to perturb $\hat{f}$:

$$f_h(g) = \sum a_{ij} (\alpha_i, h)(\beta_j, h)(\lambda_i + \nu_j, g) \qquad \text{for all } h \in H.$$

We now note that for each $h \in H$, there exists $\mu = \mu_h$ in $M(G)$ so that

$$(*) \qquad \hat{\mu}_h(\lambda_j + \nu_i) = \overline{(\alpha_i, h)(\beta_j, h)},$$

and $\sup_h \|\mu_h\|_M = b < \infty$. This follows immediately from Fact V (see Remark (i)). We proceed exactly as in 5.7.7 of [11]: $f = f_h * \mu_h$, and therefore

$$\|f\|_p \leq \|f_h\|_p \|\mu_h\|_M \leq b\|f_h\|_p .$$

That is,

$$\int_G |f(g)|^p \, dg \leq b^p \int_G \left|\sum a_{ij}(\alpha_i, h)(\beta_j, h)(\lambda_i + \nu_j, g)\right|^p dg.$$

We integrate both sides over H, interchange the order of integration, and apply the above Lemma. □

Remarks. (i) We do not have to appeal to Fact V: We can obtain (*) directly by considering the Riesz product whose transform equals $\overline{(\alpha_i, h)}$ at λ_i, and $\overline{(\beta_j, h)}$ at ν_j.

(ii) The $\Lambda(p)$ constants of $E + F$ are inherited from $S_1 + S_2$ in Γ_1. In fact, by interchanging the roles of G and H in the above proof, we see that the behavior of the $\Lambda(p)$ constants of $E + F$ is the same in all groups.

(iii) The above argument can be easily modified to prove that $E + E$ is a $\Lambda(p)$-set for all p.

B. *$E + F$ is a Rosenthal set* (see also [8, p. 251]). We note that

$$(1) \quad (c_0(E) \hat{\otimes} c_0(F))^* = \Big\{(a_{mn}) : \exists\, C > 0 \text{ so that } \Big|\sum a_{mn} x_{mn} y_n\Big| \leq C\|x\|_\infty \|y\|_\infty \text{ for all } x, y \in c_0(E),\ c_0(F) \text{ respectively}\Big\}.$$

But, $(c_0(E) \hat{\otimes} c_0(F))^* = L^\infty_{E+F}(G)\hat{}$, and since the right-hand side of (1) can be viewed as a sup-norm closure of trigonometric polynomials with spectrum in $E + F$, our assertion follows. □

C. *$E + F$ is a $4/3$-Sidon set.* By (1) above, the assertion is precisely the content of the third inequality in Theorem 1 (1) of Littlewood [7]:

$$\left(\sum\sum |a_{mn}|^{4/3}\right)^{3/4} \leq A\left(\sup_{x,y;\|x\|_\infty, \|y\|_\infty \leq 1} \left|\sum a_{mn} x_m y_n\right|\right). \quad \square$$

Littlewood observes in [7] that the 4/3 is sharp.

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